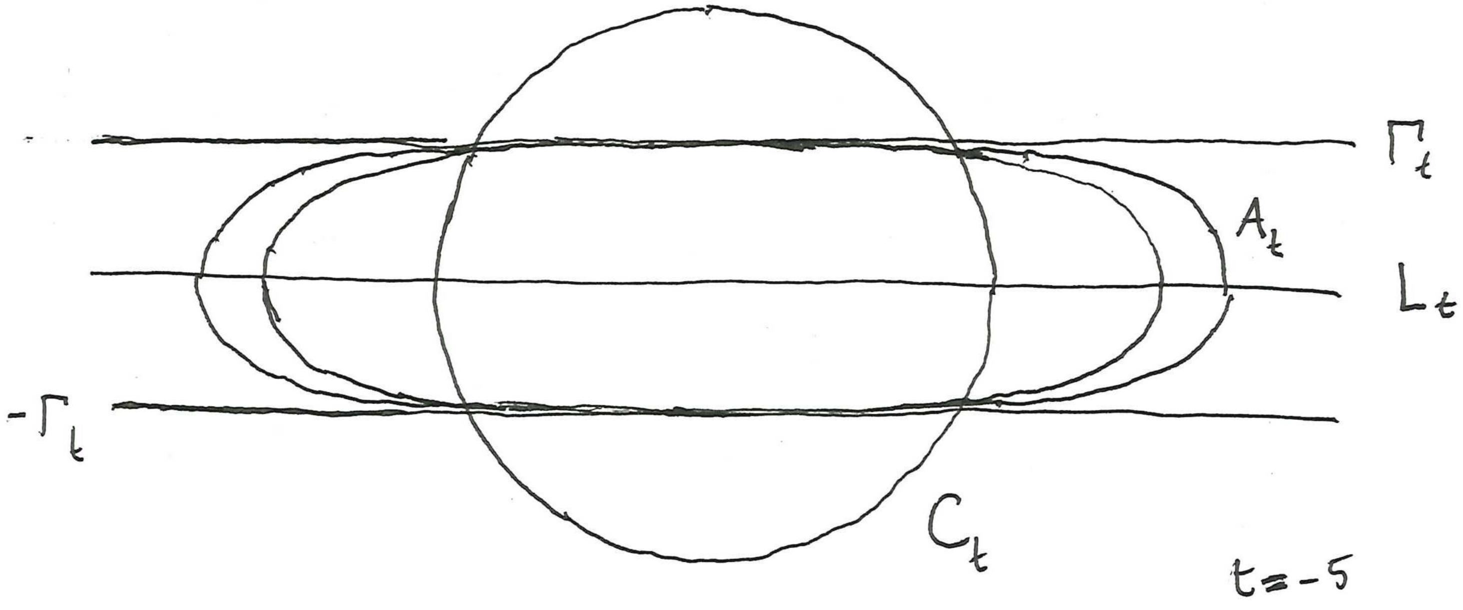
Convex Ancient Solutions to Curve Shortening Flow



CSF

$$\gamma : M^1 \times I \to \mathbb{R}^2, \quad M^1 \in \{S^1, \mathbb{R}\}$$

$$(\partial_t \gamma)^\perp = \vec{\kappa} = \gamma_{ss}$$

Ancient : $I = (-\infty, \omega)$, $\omega \in (-\infty, \infty]$

Convex : $\Gamma_t := \gamma(M^1, t) = \partial \Omega_t$, Ω_t convex (preserved if $M^1 = S^1$)

locally uniformly convex : $\kappa > 0$

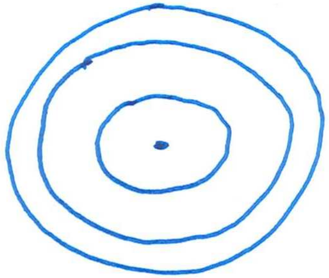
($\partial_t \kappa = \kappa_{ss} + \kappa^3$ so convex $\Rightarrow$ flat or locally uniformly convex)

Compact : $M^1 = S^1$ (bounded)

non-compact : $M^1 = \mathbb{R}$ (unbounded)

Examples

$\{S_{\sqrt{-2t}}\}_{t\in(-\infty,0)}$

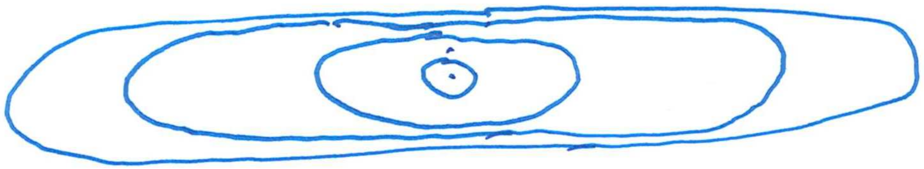


Shrinking circle

$\{L_t\}_{t\in(-\infty,\infty)}$, $L_t = \{(x,0),\ x\in\mathbb{R}\}$

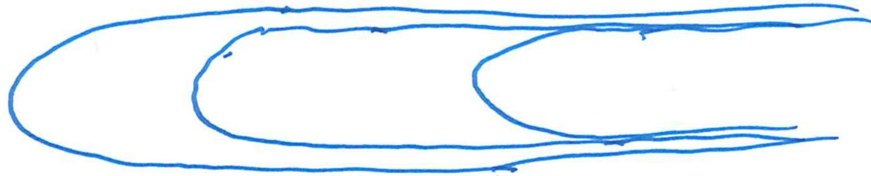
stationary line

$\{A_t\}_{t\in(-\infty,0)}$, $A_t = \{(x,y) \mid \cos x = e^t \cosh y\}$



Angenent oval

$\{G_t\}_{t\in(-\infty,\infty)}$, $G_t = \{(x, t - \log\cos x) \mid x \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$



Grim Reaper

+ space-time translations, spatial rotations, parabolic rescaling

Theorem (Daskalopoulos–Hamilton–Šešum)

The shrinking circles & Angenent ovals are the only <u>bounded</u>, <u>convex</u> ancient sol^ns to CSF in $\mathbb{R}^2$

proof: On the Angenent oval,

$$k^2 = \cos 2\theta + \coth(-2t)$$

$\Rightarrow$ $\alpha := (k^2)_\theta$ satisfies

$$\alpha_{\theta\theta} + 4\alpha \equiv 0$$

On the other hand, if

$$\alpha_{\theta\theta} + 4\alpha \equiv 0,$$

then

$$\alpha_t = k^2(\alpha_{\theta\theta} + 4\alpha)$$

$$\equiv 0$$

$\Rightarrow$ $k^2 = a(\theta) + b(t)$

$\Rightarrow$ $k^2 = b(t)$ (shrinking circle)

or $k^2 \propto \cos 2\theta + \coth(-2t)$ (Angenent oval)

(or $k^2 = a(\theta)$ (Grim Reaper))

So consider

$$I(t) := \int (\alpha_\theta^2 - 4\alpha^2)\, d\theta \;=\; -\int \frac{\alpha \alpha_t}{\kappa^2}\, d\theta$$

Observe that

$$I' = -2 \int \frac{\alpha_t^2}{\kappa^2}\, d\theta \leq 0 .$$

$\alpha_t = \kappa^2(\alpha_{\theta\theta} + 4\alpha)$.

Harnack $\Rightarrow$ $\lim_{t \to -\infty} I(t) \leq 0$

$\boxed{\kappa_t > 0}$

Gage-Hamilton $\Rightarrow$ $\lim_{t \to 0} I(t) \geq 0$

$\square$

Cf. semi-linear heat eqn (Merle-Zaag), .
2D-Ricci flow (Daskalopoulos-Hamilton-Šešum) .

Theorem (Bourni - L. - Tinaglia)

The stationary lines, shrinking circles, Grim Reapers & Angenent ovals are the only convex ancient solutions to CSF in $\mathbb{R}^2$.

Monotonicity Formula

Gaussian area: $\Theta(t) := \int_{\Gamma_t} \Phi(p,t)\, d\mathcal{H}^1(p)$,

$$\Phi(p,t) := (-4\pi t)^{-1/2} e^{\frac{|p|^2}{4t}}$$

Thm (Huisken): $\frac{d}{dt}\Theta(t) = -\int_{\Gamma_t} \left| \vec{\kappa}(p) + \frac{p^\perp}{-2t} \right|^2 \Phi(p,t)\, d\mathcal{H}^1(p)$

$$\leq 0$$

+ strict inequality unless $\vec{\kappa}(p) + \frac{p^\perp}{-2t} \equiv 0$ ~~…~~

$\iff$ shrinker, ~~…~~ $\Gamma_t = \sqrt{-t}\,\Gamma_{-1}$

(+ convexity) $\iff$ shrinking circle, stationary line, or stationary line of <u>multiplicity</u> <u>two</u>

(! classification of convex shrinkers (in $\mathbb{R}^2$))

Lemma: The blow-down, $\lim\limits_{\lambda \to 0} \{ \lambda \Gamma_{\lambda^{-2} t} \}_{t \in (-\infty, 0)}$, exists. It is either $\{ S_{\sqrt{-2t}} \}_{t \in (-\infty, 0)}$ or, up to a rotation, $\{ L_t \}_{t \in (-\infty, 0)}$ of multiplicity one or two.

proof: Convexity $\Rightarrow$ $H < C$ for all $\{\Gamma_t\}$. Thus,

(modulo limits)

$$
\begin{aligned}
-\int_a^b \int_{\lambda \Gamma_{\lambda^{-2} t}} \left| \vec{k}(p) + \frac{p^\perp}{-2t} \right|^2 d\mathcal{H}^1(p) &= H_\lambda(b) - H_\lambda(a) \\
&= H(\lambda^{-2} b) - H(\lambda^{-2} a) \\
&\to 0
\end{aligned}
$$

$\square$

* Cf. Wang

Lemma: If the blow-down is

1 $\{S_{\sqrt{-2t}}\}_{t\in(-\infty,0)}$, then $\{\Gamma_t\}$ is a shrinking circle

2 $\{L_t\}_{t\in(-\infty,0)}$ of mult. 1, then $\{\Gamma_t\}$ is a stationary line

proof: 1 $\Rightarrow$ Γ_t is bounded. The Gage-Hamilton theorem then implies that, after a space-time translation,

$$\lim_{\lambda\to\infty}\{\lambda\Gamma_{\lambda^{-2}t}\} \to \{S_{\sqrt{-2t}}\}_{t\in(-\infty,0)}.$$

But then H is constant & hence Γ_t is $\{S_{\sqrt{-2t}}\}_{t\in(-\infty,0)}$

2 similar, since the blow-up at a regular point is a stationary line.

$\square$

Harnack inequality

$k > 0 \Rightarrow$ turning angle, $\hat{\nu} = (\cos\theta, \sin\theta)$, well-defined.

Thm (Hamilton): $\partial_t \left(\sqrt{t-\alpha}\, k(\theta,t)\right) \geq 0$ ①

$\alpha = -\infty \Rightarrow \partial_t k(\theta,t) \geq 0$. ②

+ strict inequality unless $\partial_t k(\theta,t) \equiv 0$

$\Leftrightarrow$ translator, $\Gamma_t = \Gamma_0 + t\vec{e}$.

(stationary line or Grim Reaper)

(! classification of (convex) translators in $\mathbb{R}^2$)

(strict inequality in ① unless $\partial_t(\sqrt{t-\alpha}\, k) \equiv 0 \Leftrightarrow$ expander,

$\Gamma_t = \sqrt{t-\alpha}\, \Gamma_{\alpha+1}$

Theorem (Wang): If the blow-down is $\{2L_t\}_{t\in(-\infty,0)}$, then $\{\Gamma_t\}$ lies in a ~~slab~~ (parallel) slab.

Key to the proof:

Lemma: The arrival time, $u : \bigcup_t \Gamma_t \to \mathbb{R}$,

$$u(p) = t \Leftrightarrow p \in \Gamma_t,$$

is locally concave.

proof: Differentiating

$$u(\gamma(\theta,t)) = t$$

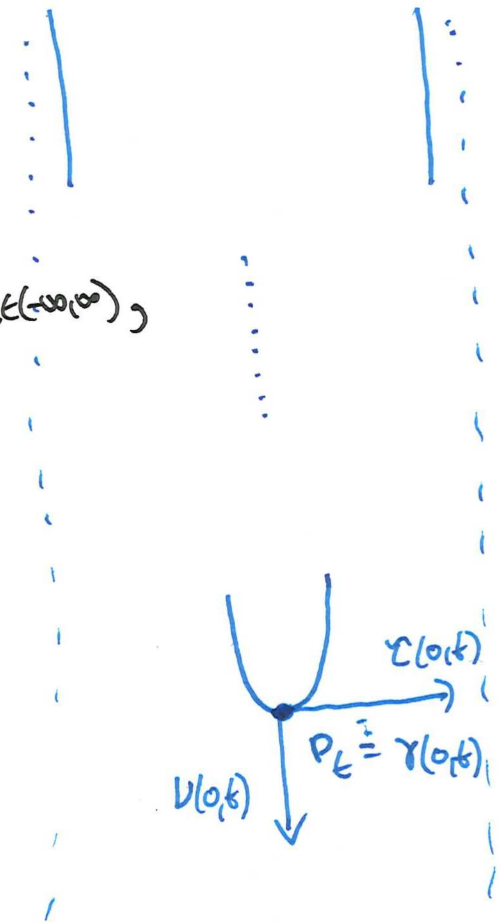
in tangential & normal directions yields

$$-D^2 u = \begin{bmatrix} 1 & 0 \\ 0 & \kappa_t/\kappa^3 \end{bmatrix} \qquad \square$$

Remains to classify <u>unbounded</u>, <u>loc. unif.</u> <u>convex</u> solutions <u>in $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}$</u> (& no thinner slab).

(Following proof also works in the bounded case).

Wlog, $\vartheta \in (-\frac{\pi}{2},\frac{\pi}{2})$, & $\omega > 0$,



Lemma: $\left\{\Gamma_{t+s} - P_s\right\}_{t\in(-\infty,-s)} \xrightarrow{C^\infty_{loc}} \left\{r\,G_{r^{-2}t}\right\}_{t\in(-\infty,\infty)}$,

where $\displaystyle r^{-1} := \lim_{s\to-\infty} K^{\infty}(0,s)$.

<u>proof</u>: • r^{-1} exists by Harnack (pos. <u>a posteriori</u>)

• Harnack $\Rightarrow$ K bounded $\Rightarrow$ <u>subseq.</u> convergence

• $K^\infty(0,t)$ constant by construction,

Harnack $\Rightarrow$ sublimit is $\{r\,G_{r^{-2}t}\}$

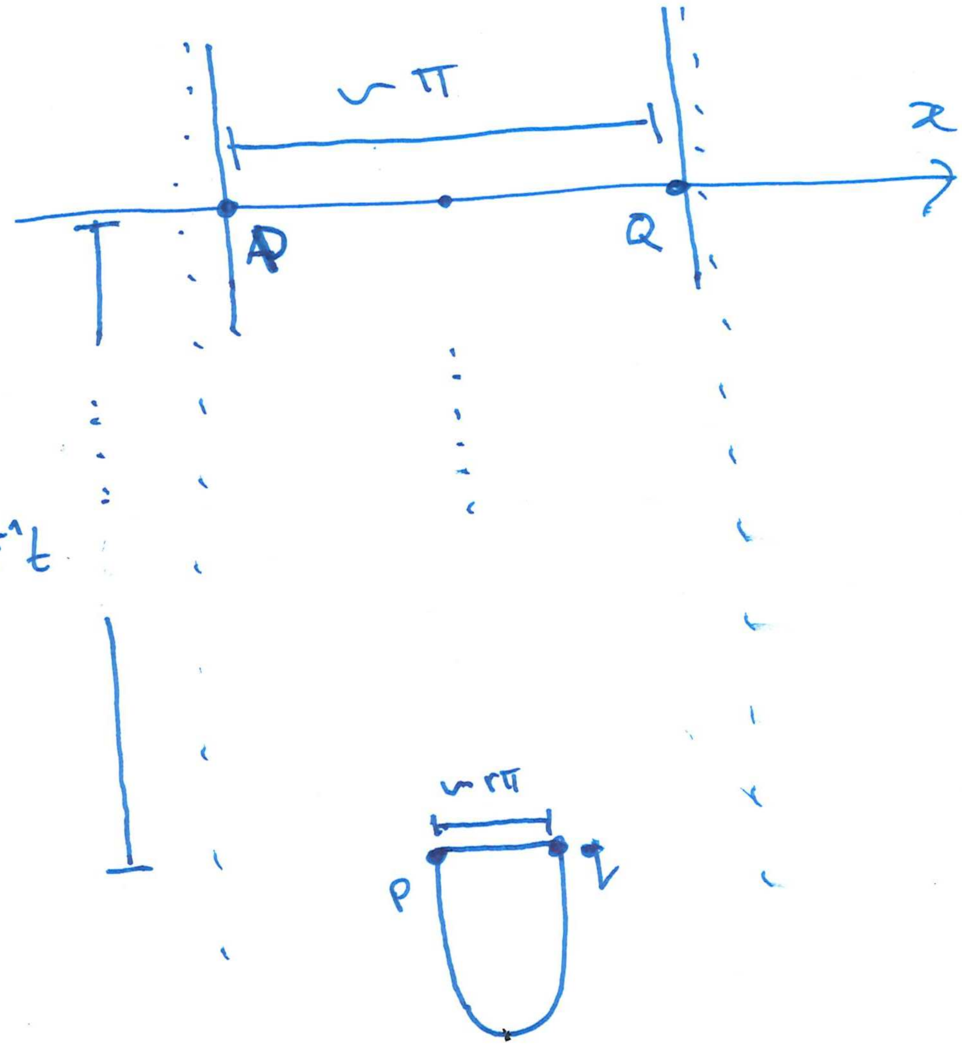
(since $\nu(0,0) = (0,-1)$). $\square$

Elementary corollary: $\bigcup_{t<0} \Omega_t = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$.

proof: use convexity & $\lim_{s\to-\infty} k(0,s) > 0$.

(• Wlog, $\{y=0\}$ supports Γ_0.)

Lemma: $r = 1$.



proof: on the one hand,

$$-\frac{d}{dt} \text{Area}(\Omega_t \cap \{y<0\}) = \int_P^Q k \, ds$$
$$< \pi$$

$\Rightarrow$ $\text{Area}(\Omega_t \cap \{y<0\}) < -\pi t$.

On the other hand,

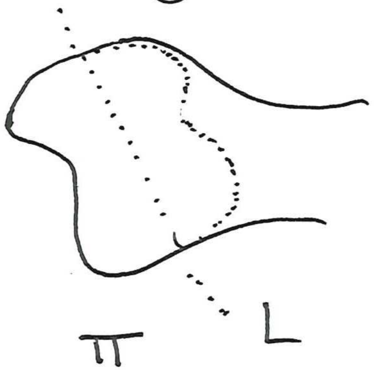
$$\text{Area}(\Omega_t \cap \{y<0\}) > \text{Area}(PQqp) \sim -\frac{\pi}{2}(1+r) \cdot r^{-1} t$$

$\square$

Alexandrov reflection

Thm (Chow–Gulliver): If Γ_{t_0} is <u>bounded</u> & reflects about the oriented line L, then Γ_t reflects about L for all $t > t_0$.

proof: Strong maximum principle + Hopf boundary point lemma. $\square$



Γ_t "reflects about L" if $R\Gamma_t \cap \Gamma_t = \phi$, where $R\Gamma_t$ is the reflection about L of $\Gamma_t \cap \Pi$, where $\partial\Pi = L$ (with orientation).

If Γ_{t_0} is unbounded, the argument still works if Γ_{t_0} "reflects about L at infinity" for all $t \geq t_0$.

Lemma: Γ_t is reflection symmetric about $\{x=0\}$ for all t.

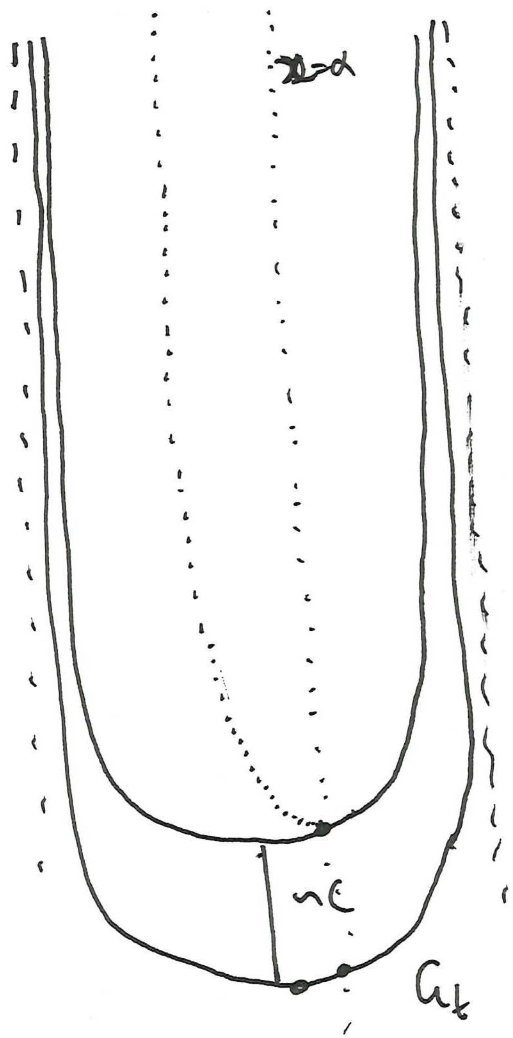
Lemma: $\lim\limits_{t\to-\infty} (\ell(t)+t) \geq 0$, where $\ell(t) = \langle \gamma(0,t), \nu(0,t) \rangle$.

proof: Harnack inequality $\Rightarrow$ $\frac{d}{dt}(\ell(t)+t) \leq 0$

(possibly ∞)

Suppose, "wlog", that $C := \lim\limits_{t\to-\infty} (\ell(t)+t) < 0$

Then, for any $\alpha > 0$, the reflection of Γ_t about $\{x = \alpha\}$ lies above C_t for all t suff. small.



since $C < 0$ & $\ell(t)+t$ is monotone, "contact" can never occur at the plane of symmetry.

Alexandrov refl. princ. then implies that the reflection lies above C_t $\forall\, t > t_\alpha$

$\alpha \to 0 \Rightarrow \Gamma_t \cap C_t = \emptyset$ ↯ $(t=0)$ $\square$

Given $\tau > 0$, set $\Gamma^{\tau}_t := \Gamma_{t+\tau}$. Then

$$\lim_{t \to -\infty} \left(\ell(t+\tau) + t \right) > 0$$

& hence, by the preceding argument,

$$\Gamma_{t+\tau} \text{ lies above } C_t \quad \text{for all } t$$

Taking $\tau \to 0$,

$$\Gamma_t \text{ lies above } C_t . \quad \text{for all } t$$

Since $0 \in \Gamma_0 \cap C_0$, the strong maximum principle implies

$$\Gamma_t = C_t \quad \text{for all } t. \qquad \square$$

Remarks:

- Similar argument $\Rightarrow$ $\{A_t\}_{t\in(-\infty,0)}$ is the only bounded example in a slab.
- Alexandrov reflection was used to obtain uniqueness of the "shrinking parallel" in S^2 by Bryan & Louie.
- Wang implicitly assumed compact time slices. This was required in his proof (missing some details) that the arrival time is concave (concavity maximum principle (≠ Trudinger)). We remove this assumption in all dimensions in another paper (but require $\sup_t |A| < \infty$ when $n > 1$).
- "scaling" arguments + monotonicity formula very useful for entire solutions (but not for non-entire)
- "translating" + Harnack very useful for non-entire solutions (but apparently not for entire ones).

Higher dimensions (slabs)

Let $\{M_t\}_{t\in(-\infty,\omega)}$ be a convex ancient solution in $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}^n$.
with $\boxed{\sup_{M_t}|A|<\infty \text{ for all } t}$. ← superfluous?

By the Harnack inequality, the squash-down

$$M_* := \lim_{t\to-\infty}\frac{1}{-t}M_t$$

exists. (It is a convex body in $\{0\}\times\mathbb{R}^n$).

Theorem (BLT): M_* circumscribes $\{0\}\times S^{n-1}$.

(* Use this to obtain reflection symmetry).

Conjecture (BLT): For any convex body P in $\{0\}\times\mathbb{R}^n$ which circumscribes $\{0\}\times S^{n-1}$, there exists an ancient sol$^{\underline{n}}$ to MCF in $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}^n$ with squash-down P.

We were able to prove existence when P is

- a regular (unbounded) polytope
  (the unbounded P yield translators)

- an (unbounded) simplex
  (the unbounded P yield translators)

- a truncated regular cone
  (these examples are eternal & nontranslating, which disproves a conjecture of White).

(*) Construction relies heavily on the existence (BLT) of a rotationally symmetric example (& its properties).

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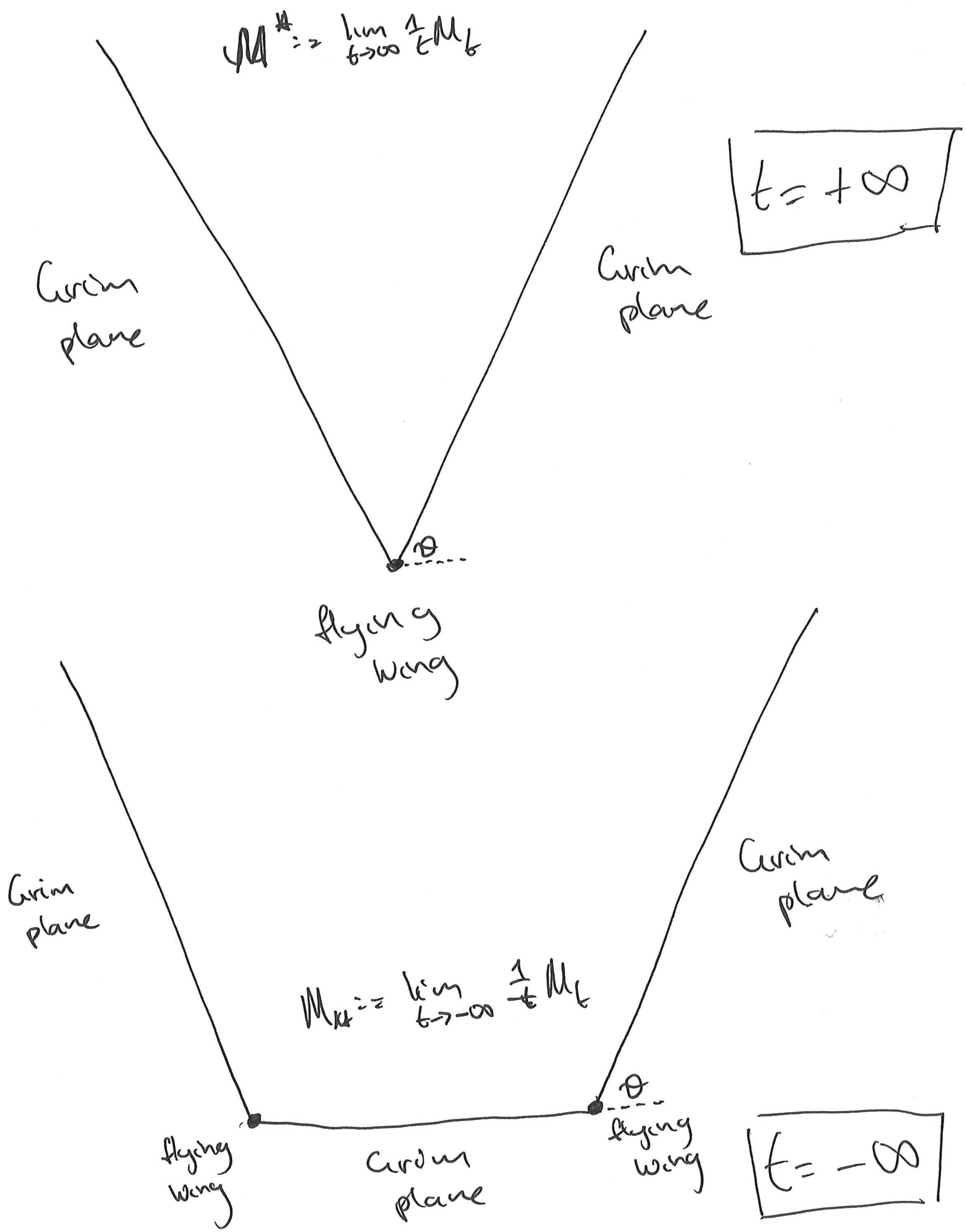

* total exterior angles preserved.