MATH704 DG Sem 2, 2018: Assignment 01

1 Question 01: Tubular Neighbourhood

Let $\gamma : (a, b) \to \mathbb{R}^2$ be a *regular* smooth curve.

1. Show that for every $t_0 \in (a, b)$, there is a local, *tubular neighbourhood* of γ. That is, there is $\epsilon > 0$ and a $\delta > 0$ such that

$$\Gamma(t, u) = \gamma(t) + uN(t), \quad t \in (t_0 - \delta, t_0 + \delta), \quad u \in (-\epsilon, \epsilon)$$

   is diffeomorphic with an open set $V \subset \mathbb{R}^2$ containing $\{\gamma(t) : t \in (t_0 - \delta, t_0 + \delta)\}$.

2. Choose one of the following two options:

   (a) Show that for each fixed u, the curves $\gamma_u(t) = \Gamma(t, u)$ are *parallel* in the sense that $T_u(t) = T_0(t) = T_\gamma(t)$ for each $u \in (-\epsilon, \epsilon)$ and $t \in (t_0 - \delta, t_0 + \delta)$. Give an example of a curve for which the map Γ is a diffeomorphism for all $\epsilon > 0$. Give an example where this is false.

   (b) Pick any *interesting* curve γ and plot the parallel curves γ_u described in the other option and showing parallel tangent vectors. Plot a curve where Γ is a diffeomorphism for all ϵ including showing the normal, and plot a curve where it is false showing points where $\Gamma(t_1, u_1) = \Gamma(t_2, u_2)$ for some $(t_1, u_1) \neq (t_2, u_2)$.

   Please by sure to provide a copy of your code electronically so that I can also run the code.

3. Show that for any c, d with $a < c < d < b$, there exists a *global* tubular neighbourhood of γ restricted to $[c, d]$. That is, there exists an $\epsilon > 0$

such that for $t \in (c,d)$ and $u \in (-\epsilon, \epsilon)$, Γ is a diffeomorphism onto an open set $V \subset \mathbb{R}^2$ containing $\{\gamma(t) : t \in (c,d)\}$.

Hint: You may assume the following *compactness result*: Since the interval $[c,d]$ is closed and bounded there are finitely many intervals $\{I_i = (a_i, b_i)\}_{i=1}^N$ such that $[c,d] \subset \cup_{i=1}^N I_i$ and for which $\Gamma_i = \Gamma|_{I_i}$ is a diffeomorphism onto an open set for $u =\in (-\epsilon_i, \epsilon_i)$.

Use the compactness result to get a *uniform* $\epsilon > 0$ (i.e. independent of $t \in [c,d]$) for which $\Gamma|_{I_i}$ is a diffeomorphism.

Now show that Γ is in fact *injective* for $t \in (c,d)$ and $u \in (-\epsilon, \epsilon)$ and argue that this proves Γ has a well defined smooth inverse.

2 Question 02: Curvature determines the curve

Let $\kappa : (a,b) \to \mathbb{R}$ be any smooth function and let $\theta = \int \kappa$ be any anti-derivative. Fix the orientation so that $N = J(T)$ where J is *counter-clockwise* rotation by $\pi/2$.

1. Show that
$$\gamma(s) = \left(\int_a^s \cos(\theta(t))dt, \int_a^s \sin(\theta(t))dt \right)$$
is a regular curve with curvature equal to κ.

2. Conversely show that if γ is any other curve with curvature equal to κ then there is a $\theta_0 \in \mathbb{R}$ and a $p \in \mathbb{R}^2$ such that
$$\gamma(s) = \left(\int_a^s \cos(\theta(t) + \theta_0)dt, \int_a^s \sin(\theta(t) + \theta_0)dt \right) + p$$
where θ is the anti-derivative chosen in the previous part.

   Hint: you may assume that the function φ determined (up to addition of integer multiples of 2π) by $T(s) = (\cos\varphi(s), \sin\varphi(s))$ is smooth.

   If you're feeling especially motivated, consider what happens under *reflections*.

3. Choose one of the following two options:

(a) Show that if $\varphi(s)$ denotes the angle the tangent $T(s)$ makes with the x-axis, then $\varphi(s) = \theta(s) + \theta_0$ for some $\theta_0 \in \mathbb{R}$. Show moreover that the angle between $T(s_1)$ and $T(s_2)$ satisfies

$$\sphericalangle T(s_1), T(s_2) = \int_{s_1}^{s_2} \kappa ds.$$

This explains the phrase *Turning Tangents*.

(b) Pick any *interesting* curve γ. Write a program that takes as input t_1, t_2 and outputs the angle $\sphericalangle T(s_1), T(s_2)$ and $\int_{s_1}^{s_2} \kappa ds$. They should be equal!

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3 Question 03: Space Curves

Let $\gamma : (a, b) \to \mathbb{R}^3$ be a regular, smooth space curve.

1. Show that γ lies in a plane if and only if the torsion $\tau \equiv 0$.

2. Define the curve

$$\gamma(t) = \begin{cases} (t, 0, e^{-1/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t < 0 \\ (0, 0, 0), & t = 0. \end{cases}$$

You may assume the result that the function

$$f(t) = \begin{cases} e^{-1/t^2}, & t \geq 0 \\ 0, & t \leq 0 \end{cases}$$

is smooth and such that $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Moreover,

$$f^{(k)}(t) = p_k(t^{-1}) f(t)$$

where p_k is a polynomial of degree $3k$ with $p_k(0) = 0$.

(a) Show that γ is a regular smooth curve for all t.

(b) Show that if $t \neq 0, \pm\sqrt{2/3}$, then $\kappa(t) \neq 0$. Show also that $\kappa(0) = 0$.

Hint: Feel free to lookup formulas for κ in terms of an arbitrary parameter t. You'll need this since the parametrisation given is not by arc length. A very useful formula is $\partial_s = (1/v)\partial_t$ where $v = |\gamma'|$ from which an expression for κ is readily obtained. You may also find it convenient to express the various quantities in terms of f and it's derivatives rather than explicitly writing everything out.

3. Let γ be the same curve as in the previous question.

(a) Show that the normal is discontinuous at $t = 0$ by showing that

$$\lim_{t\to 0^+} N(t) = (0, 0, 1)$$

while

$$\lim_{t\to 0^-} N(t) = (0, 1, 0).$$

(b) Show that $\tau \equiv 0$ for $t \neq 0, \pm\sqrt{2/3}$ so that defining $\tau \equiv 0$ for all t gives a continuous torsion, yet γ does not lie in a plane.