MATH704 DG Sem 2, 2018: Assignment 03

Instructions: - Due 9th November

Your grade will be determined from your 3 best answers. Each question has three parts worth 5 points each giving a maximum of 45 points in total. Feel free to turn in answers for all four questions, but only the three best will count.

1 Question 01: The Gauss Map of a Closed Surface

A closed surface $S \subseteq \mathbb{R}^3$ is a regular surface such that

1. S is a closed subset of $\mathbb{R}^3$,
2. S is bounded: there exists an $R > 0$ such that

$$S \subseteq B_R(0) = \{x^2 + y^2 + z^2 \leq R^2\}.$$

A plane $P \subset \mathbb{R}^3$ divides $\mathbb{R}^3$ into two *sides*: $H^{\pm} = \{x \in \mathbb{R}^3 : \pm \langle \mathbf{n}, x - x_0 \rangle > 0\}$ where $x_0 \in P$ is any point in P, and $\mathbf{n}$ is the normal to P. A set S lies on one side of P if $S \subseteq H^+$ or $S \subseteq H^-$.

Prove that the Gauss map of a closed, oriented surface (compact, no boundary) S is surjective as follows:

1. Suppose there is a plane P intersecting S at x_0 such that in an open neighbourhood $V \subset S$ of x_0, S lies on one side of P. Prove that P is the tangent plane to S at x_0.

   Hint: In a local parametrisation $\phi : U \to V$, show the function $f(u, v) = \langle \phi(u, v) - x_0, \mathbf{n} \rangle$ has a local minimum at $(u_0, v_0) = \phi^{-1}(x_0)$. Hence the first derivative test implies that $\frac{\partial}{\partial u} f = \frac{\partial}{\partial v} f = 0$. Now what are the coordinate tangent vectors at (u_0, v_0)?

2. Using the definition of closed surface above, show that for any unit vector $\mathbf{n} \in \mathbb{R}^3$, there exists a plane P with unit normal vector $\mathbf{n}$ such that $P \cap S = \emptyset$.

   Now consider the map

   $$\Phi(t, Z) = Z + t\,\mathbf{n}, t \in \mathbb{R}, Z \in P.$$

   Then for each $t_0 \in \mathbb{R}$, $P(t) = \{\Phi(t_0, Z) : Z \in P\}$ is a plane and $P(0) = P$.

   Show that there exists a $t_0 \in \mathbb{R}$ such that $P(t_0) \cap S \neq \emptyset$ and S lies on one side of $P(t_0)$.

3. Using the previous parts show that given any unit vector $\mathbf{n}$, there is a point $x_0 \in S$ such that the unit normal $N(x) = \mathbf{n}$, and hence the Gauss map is surjective.

2 Question 02: Surfaces of Revolution

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly positive function with continuous second derivative and let S be the surface of revolution parameterized locally by

$$\varphi(z, \theta) = (f(z)\cos\theta, f(z)\sin\theta, z).$$

For all the following calculations, leave your answer in terms of f, f', f''.

Recall that the matrix representation of g in these coordinates is

$$g = \begin{pmatrix} 1+(f')^2 & 0 \\ 0 & f^2 \end{pmatrix}.$$

1. Show that the matrix representation of the second fundamental form A in these coordinates is

$$A = \pm\frac{1}{\sqrt{1+(f')^2}}\begin{pmatrix} -f'' & 0 \\ 0 & f \end{pmatrix}$$

and that the matrix representation of dN is

$$dN = \pm\frac{1}{\sqrt{1+(f')^2}}\begin{pmatrix} \frac{f''}{1+(f')^2} & 0 \\ 0 & \frac{-1}{f} \end{pmatrix}.$$

where $\pm$ depends on your chosen orientation.

2. Show that $(1,0)$ and $(0,1)$ are eigenvectors of dN and show that the corresponding eigenvalues are

$$k_1 = \frac{f''}{(1+(f')^2)^{3/2}}, \quad k_2 = \frac{-1}{f\sqrt{1+(f')^2}}.$$

3. Calculate H, K and show that $K \equiv 0$ if and only if $f(z) = az + b$ for some $a, b \in \mathbb{R}$.

3 Question 03: The Sphere

1. On $\mathbb{S}^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\}$ let $N = (0, \cdots, 0, 1)$ and $S = (0, \cdots, 0, -1)$ denote the north and south poles respectively. Let π_N and π_S denote stereographic projection based at N and S respectively.

   Show that $\pi_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ and $\pi_S : \mathbb{S}^n \setminus \{S\} \to \mathbb{R}^n$ are bijections.

2. Show that the transition map $\tau_{NS} = \pi_N \circ \pi_S^{-1}$ maps $\mathbb{R}^n \setminus \{0\}$ diffeomorphicaly with itself.

3. Show that the metric on $\mathbb{R}^n$ in coordinates $\pi_N^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is

$$g_N(x) = \varphi(x)\delta$$

where δ is the usual Euclidean metric and

$$\varphi(x) = \frac{4}{(1 + \sum_{i=1}^n (x_i)^2)^2}.$$

Remark: A metric of the form $\varphi\delta$ is called *conformal* to δ. In this case, since δ is the Euclidean metric which is *flat*, g_N is called *conformally flat*. Since the sphere is covered by the two open sets $\mathbb{S}^n \setminus \{N\}$ and $\mathbb{S}^n \setminus \{S\}$ on which it is conformally flat, the spherical metric is *locally conformally flat*. It is however, not globally conformally flat since a basic result in topology says that the sphere is not homeomorphic to any Euclidean space.

4 Question 04: Projective Space

1. Let $\mathbb{RP}^n$ denote the real projective space of dimension n. Show that for each $i = 1, \dots, n+1$ the maps
$$\varphi_i : [V] \in U_i \mapsto \frac{1}{V_i}\hat{V}_i \in \mathbb{R}^n$$
are *well defined bijections* where
$$U_i = \{[(v_1, \dots, v_{n+1})] \in \mathbb{RP}^n : v_i \neq 0\}$$
and
$$\hat{V}_i = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}) \in \mathbb{R}^n$$
denotes the n-vector obtained from the $(n+1)$-vector V by removing the i'th entry. Also show that the sets U_i, $i = 1, \dots, n+1$ cover $\mathbb{RP}^n$.

   Remark: The maps $\varphi_i : U_i \to \mathbb{R}^n$ are called *affine charts*.

2. Show the transition map
$$\tau_{12} = \varphi_1 \circ \varphi_2^{-1}$$
is a diffeomorphism of the open set $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \neq 0\}$ with the open set $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_2 \neq 0\}$.

   Remark: All the transition maps τ_{ij} with $i \neq j$ are of essentially the same form, just with i swapped with 1 and j swapped with \2\). Thus all the transition maps τ_{ij} are diffeomorphisms.

3. Show that the map
$$\pi : V \in \mathbb{S}^n \to [V] \in \mathbb{RP}^n$$
is smooth. That is, with respect to the stereographic charts for $\mathbb{S}^n$ and affine charts for $\mathbb{RP}^n$, we have
$$\pi_i \circ \pi \circ \pi_Z^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$
is smooth where $i = 1, \cdots, n+1$ and $Z = N, S$. For the purposes of this assignment, you may just show it for $i = 1$ and $Z = N$. The other cases are similar.

   Show also that for every $[V] \in \mathbb{RP}^n$, $\pi^{-1}([V]) = \left\{ \frac{V}{\|V\|}, \frac{-V}{\|V\|} \right\}$ consists of precisely two points.

   Remark: One can also show that $d\pi$ is an isomorphism everywhere and so $\mathbb{S}^n$ and $\mathbb{RP}^n$ are locally diffeomorphic but not globally diffeomorphic giving us a counter example to the global inverse function theorem. In this case, $\mathbb{S}^n$ is the *double cover* of $\mathbb{RP}^n$ and $\mathbb{S}^n$ is orientable, while $\mathbb{RP}^n$ is not.