Differential

$$df : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \; m\left\{ \underbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}}_{n} \right. \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix}$$

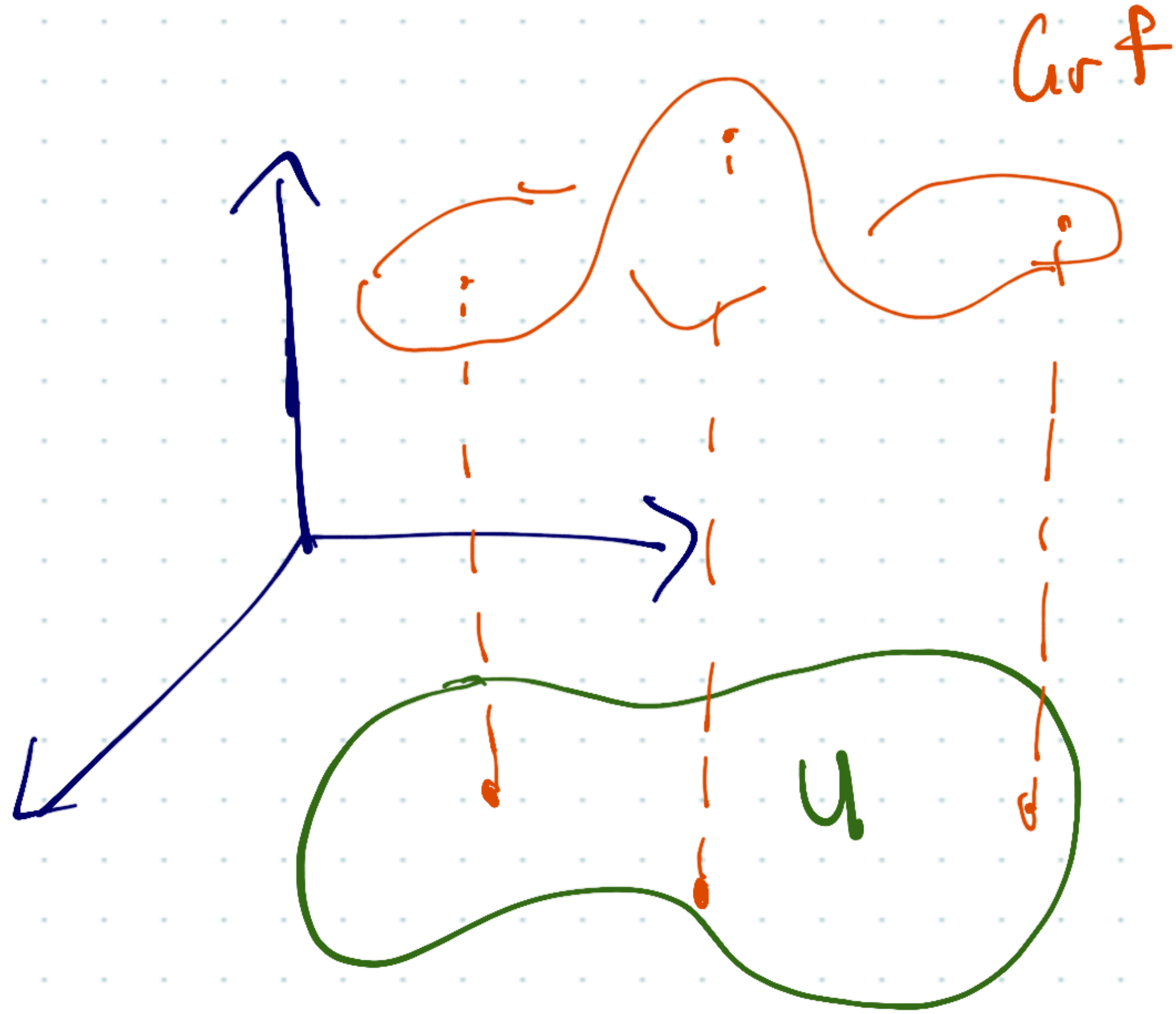
$$\partial_1 \; \partial_2 \cdots \partial_n$$

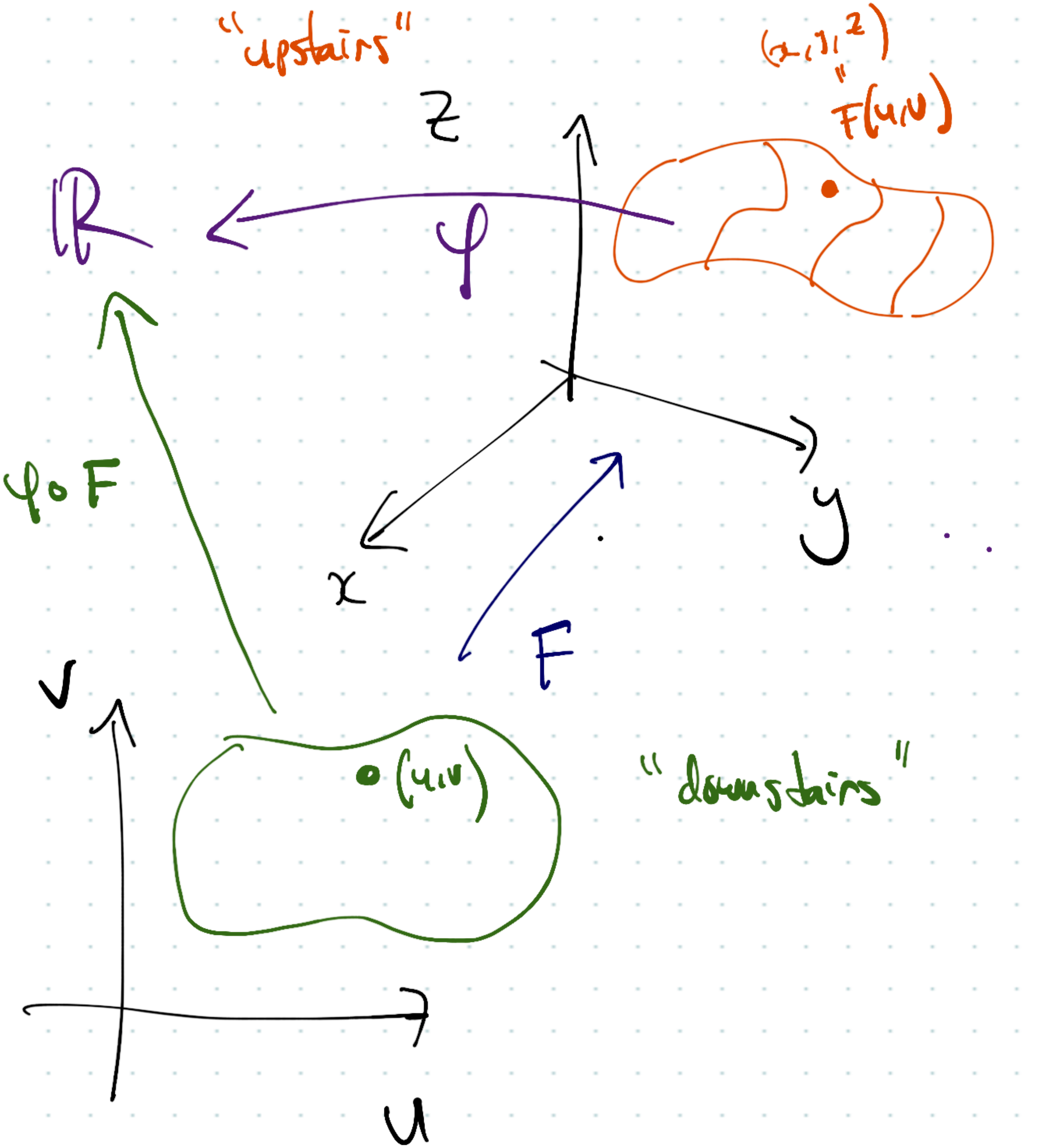
$$df_{\underset{\text{col}}{i}\,\underset{\text{row}}{j}} = \partial_i f^j \qquad \begin{array}{l} 1 \le i \le n \\ 1 \le j \le m \end{array}$$

Second Derivative

$$df = (\partial_1 f \;\cdots\; \partial_n f)$$

$$d^2 f_{ij} = \partial_i \, df_j$$
$$= \partial_i \partial_j f$$

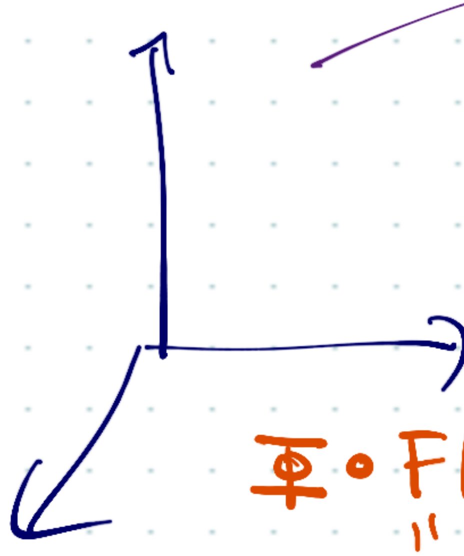

Gr f
U



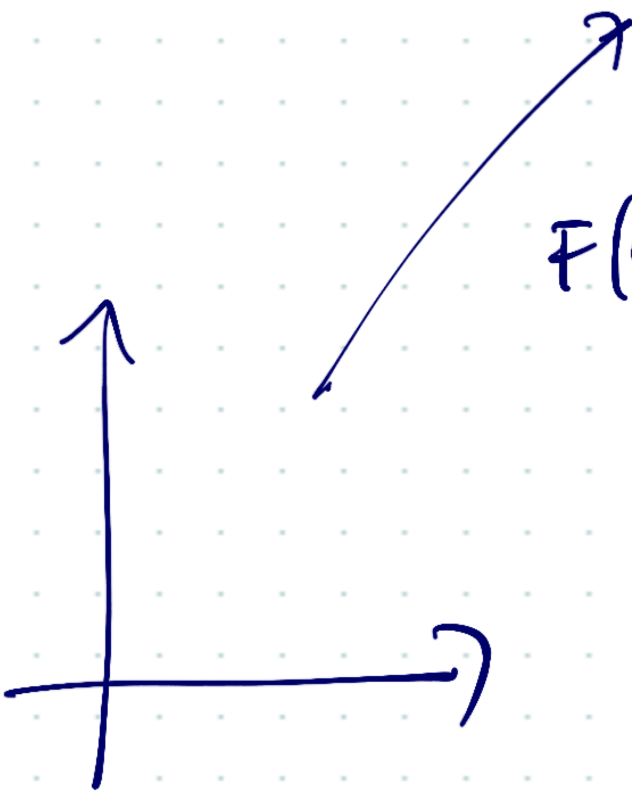
"upstairs"
(x,y,z)
"
F(u,v)
z
ℝ
φ
x
y
φ∘F
F
v
(u,v)
"downstairs"
u

Examples

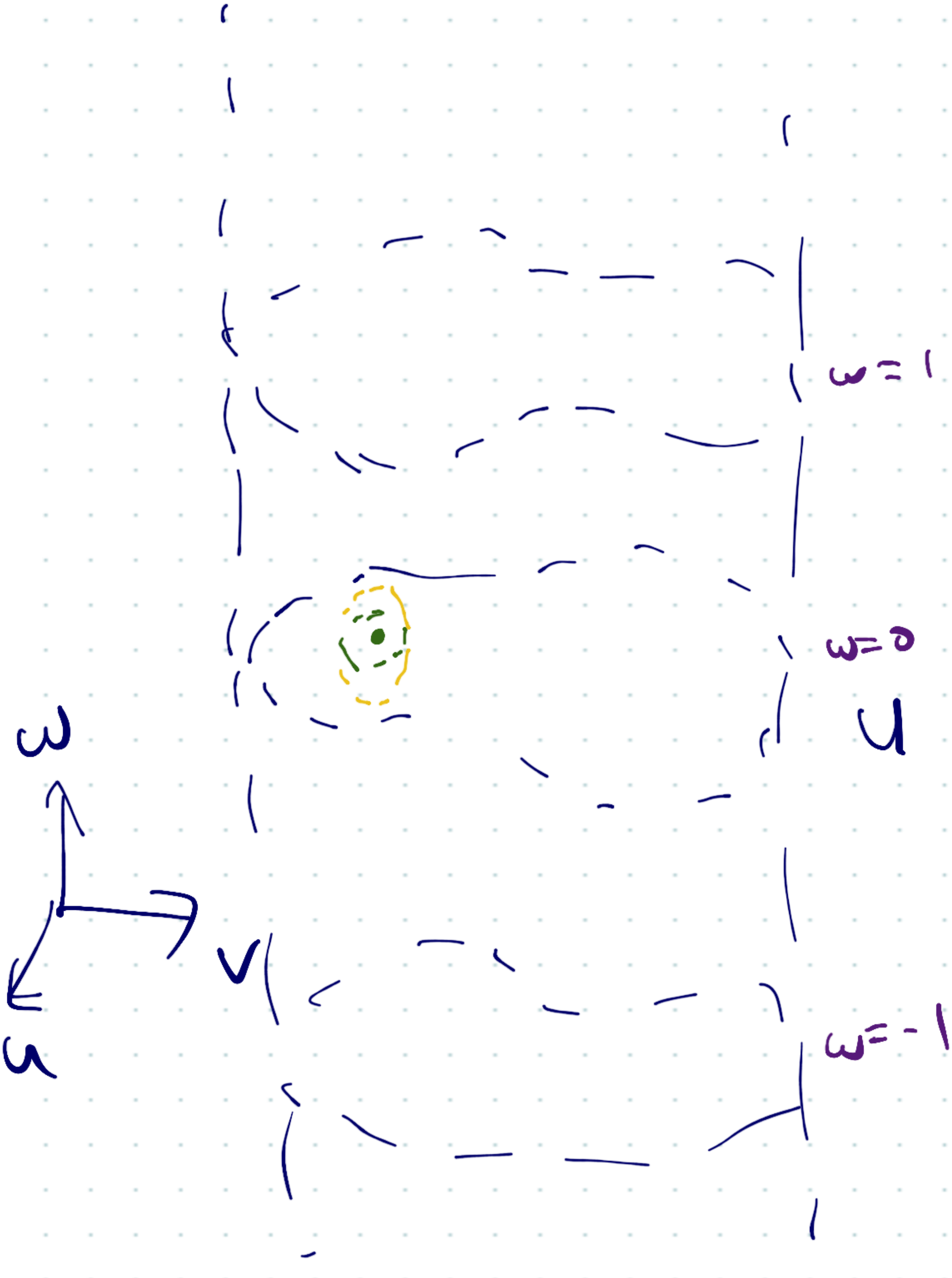
$$\Phi(x,y,z) = e^{z+y^2}$$



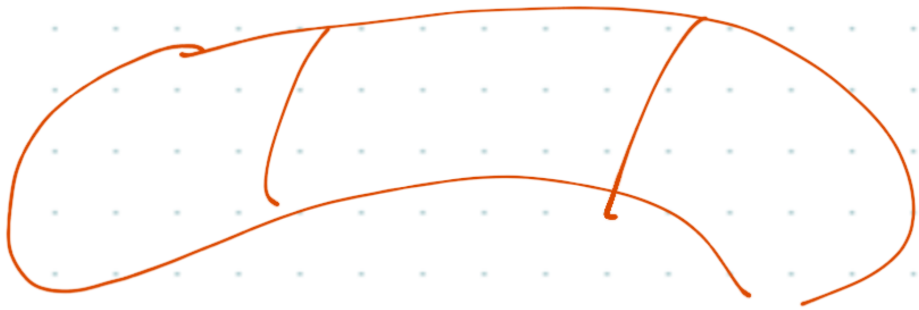
$$\Phi \circ F(u,v) = e^{u^2\cos v + v^2}$$



$$F(u,v) = \left(u, v, u^2\cos v\right)$$

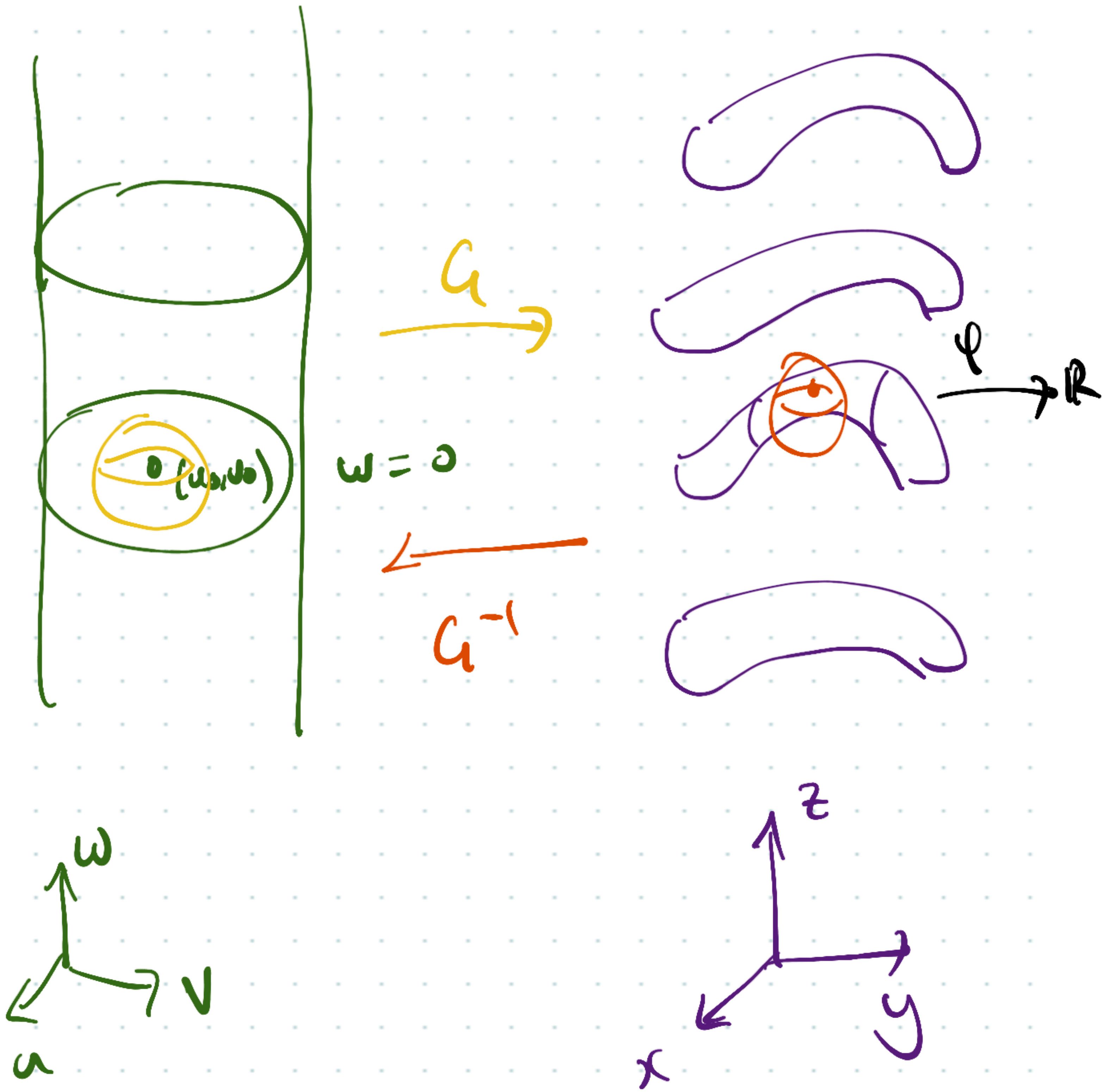

w=1
w=0
u
w=-1
w
v
u

$w = 1$



$w = 0$

$w = -1$



(u_0, v_0)
G
$w = 0$
G^{-1}
φ
$\mathbb{R}$
w
v
u
z
y
x

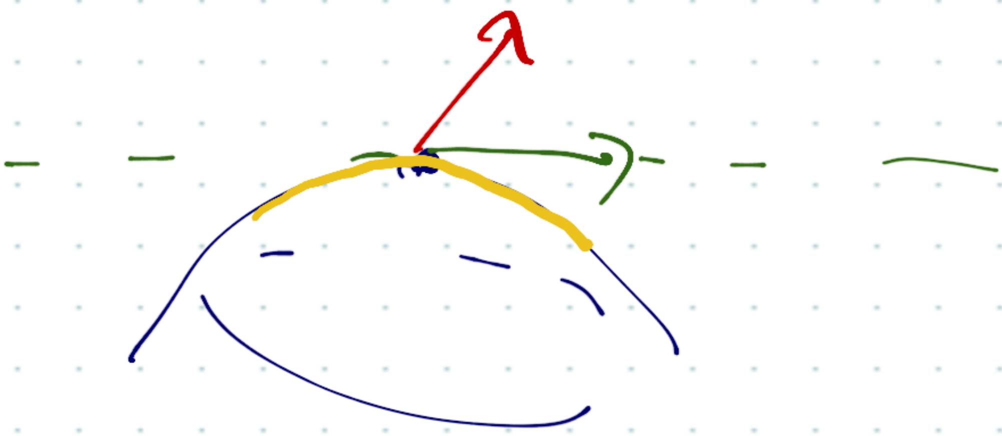
$(x,y,z) \in \operatorname{Gr} f$

$\exists!\ (u,v)$ s.t. $(x,y,z) = F(u,v)$
$= G(u,v,0)$

since $G(u,v,\omega) = F(u,v) + (0,0,\omega)$

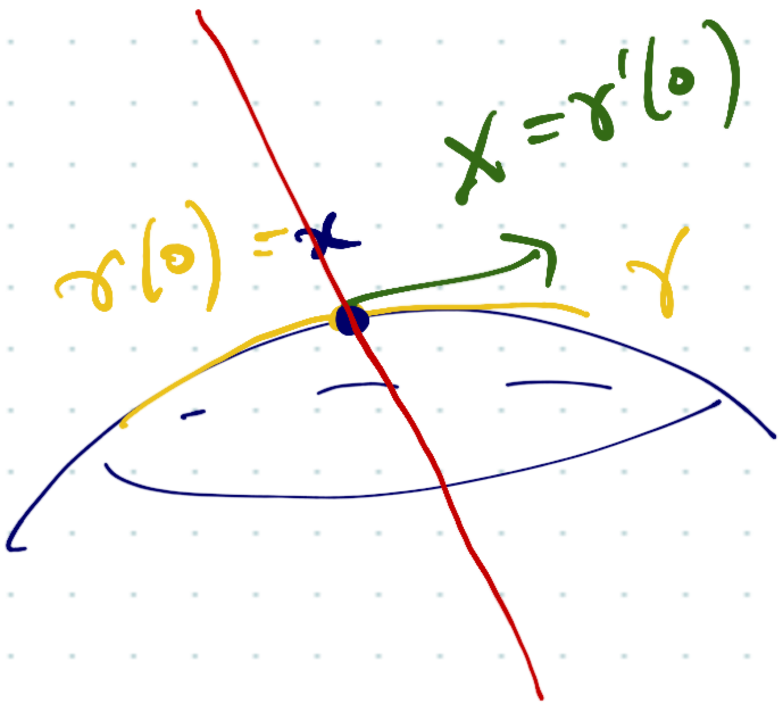
Then $G^{-1}(x,y,z) = G^{-1}(G(u,v,0))$
$= (u,v,0)$

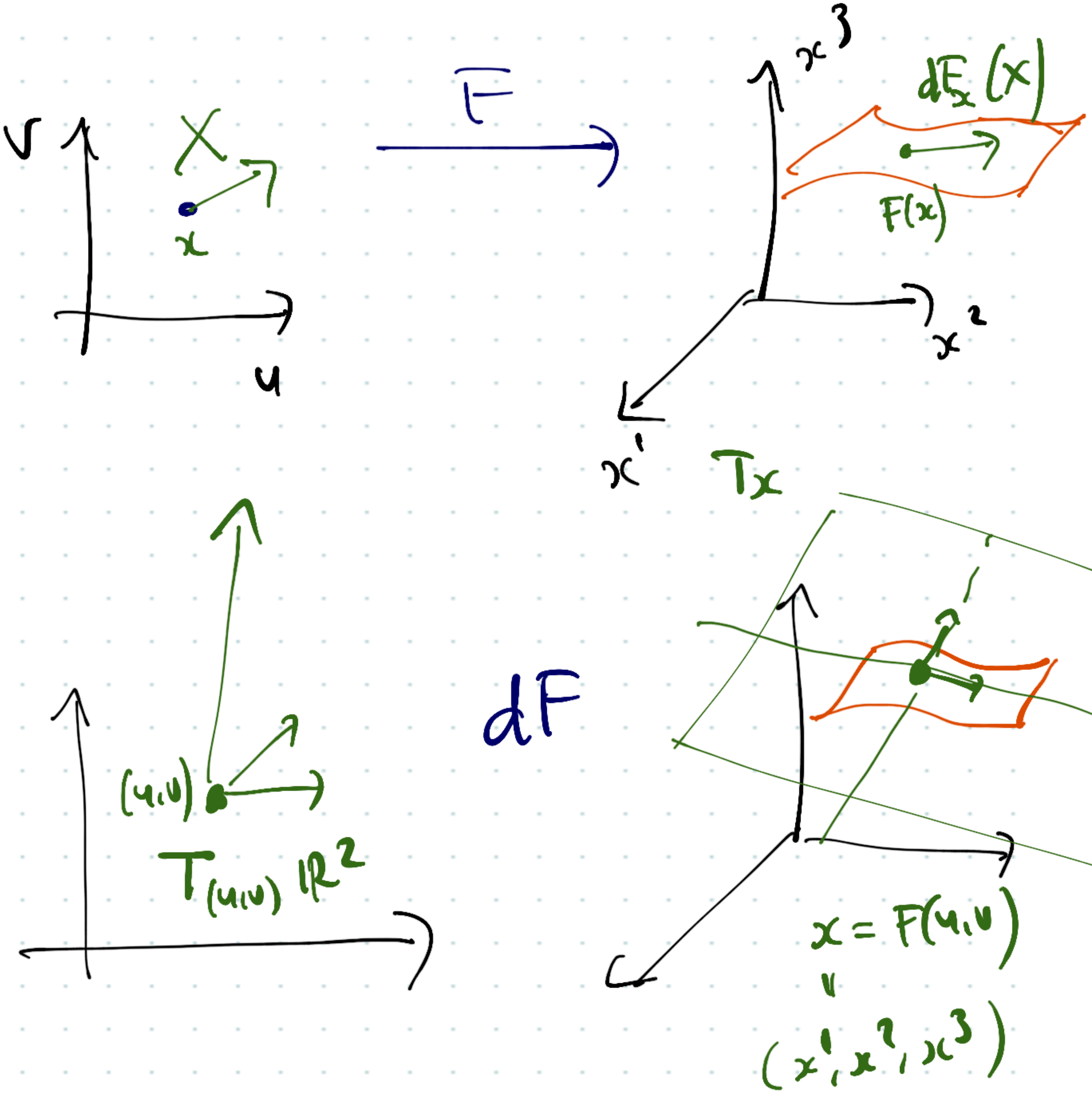
$$\Phi\big|_{\operatorname{Gr} f}^{V}(x,y,z) = \varphi \circ \bar{F} \circ G^{-1}(x,y,z)$$
$$= \varphi \circ \bar{F}(u,v,0)$$
$$= \varphi \circ F(u,v)$$
$$= \varphi(x,y,z)$$



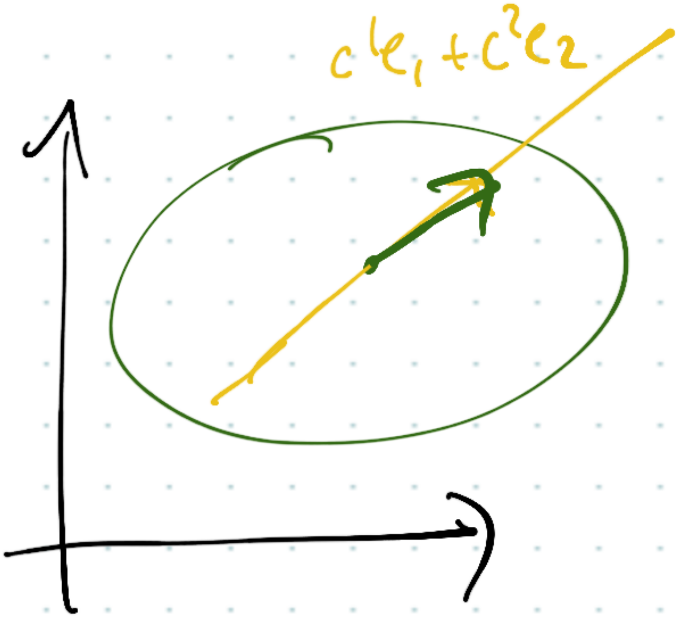
$\varphi \longrightarrow \mathbb{R}$

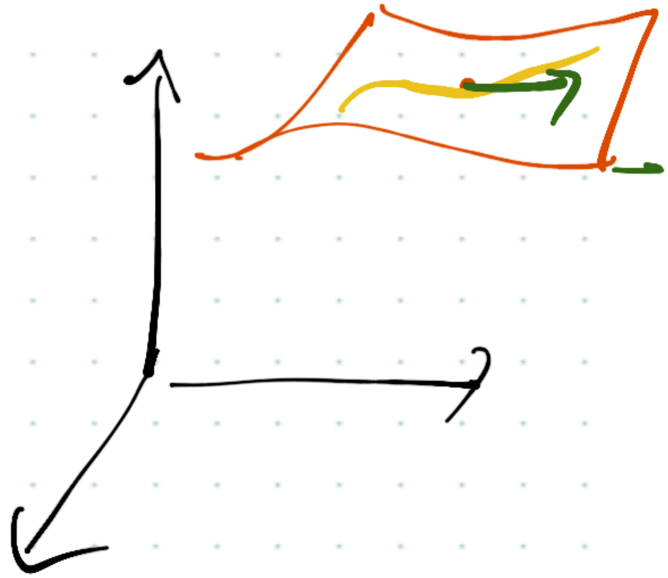
not allowed





v
u
X
x
F
x^3
x^2
x^1
dF_x(X)
F(x)
T_x
(u,v)
T_(u,v) ℝ^2
dF
x = F(u,v)
(x^1, x^2, x^3)





$$F(u,v) = (u, v, f(u,v))$$

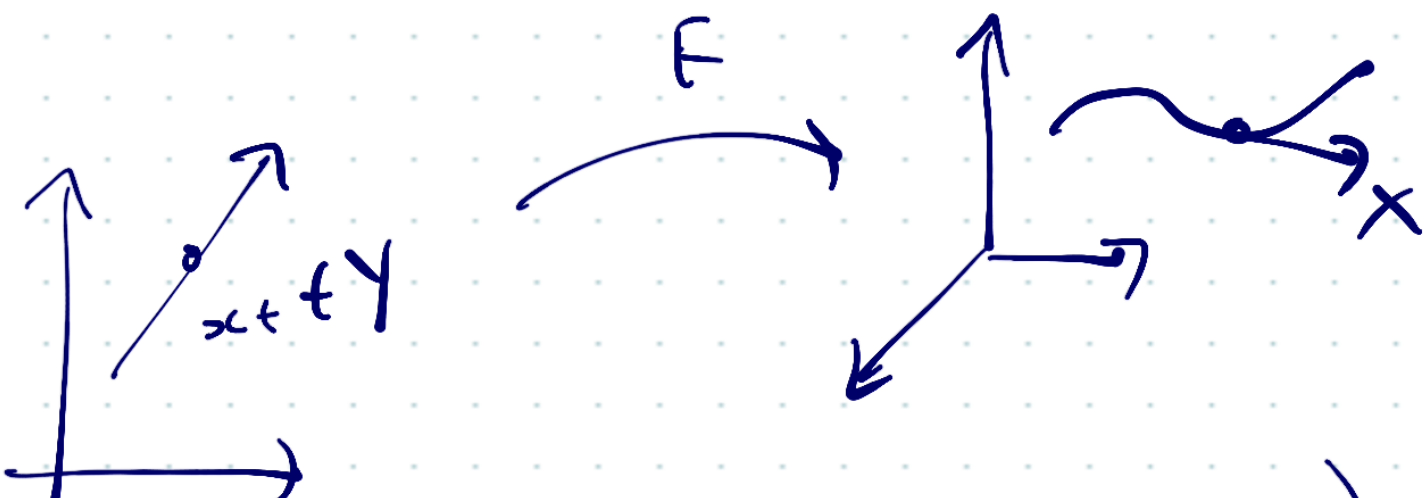
$$\partial_u F = (1, 0, \partial_u f)$$
$$= e_1 + \partial_u f e_3$$

$$dF(\mathbb{R}^2) = c^1 dF(e_1) + c^2 dF(e_2)$$
$$= c^1 \partial_u F + c^2 \partial_v F$$

if $\quad X = dF \cdot Y$

then

$$X = \frac{d}{dt}\Big|_{t=0} F(x + tY)$$



$$\underbrace{d\varphi(X)}_{\text{dfn } 2} = d\varphi(dF(Y))$$

$$= \underbrace{d(\varphi \circ F) \cdot Y}_{\text{dfn } 1}$$

$$\psi \circ F = \Phi \circ F$$

$$d(\psi \circ F)(Y)$$

$$\|$$

$$d(\Phi \circ F)(Y)$$

$$\|$$

$$d\Phi \cdot dF(Y)$$

$$d\Phi_x \cdot \gamma'(0)$$

$$\|$$

$$d\Phi \cdot X = \frac{d}{dt}\Big|_{t=0} \Phi(\gamma(t))$$

where

$\gamma(0) = x$

$\gamma'(0) = X$

$$X = dF(Y)$$

$$df(X) = \frac{d}{dt}\Big|_{t=0} f(\gamma(t))$$

$$\gamma(0) = x$$

$$\gamma'(0) = X$$

$$= df_{\gamma(0)} \cdot \gamma'(0)$$

by chain rule

$$\|$$

$$df_x \cdot X$$