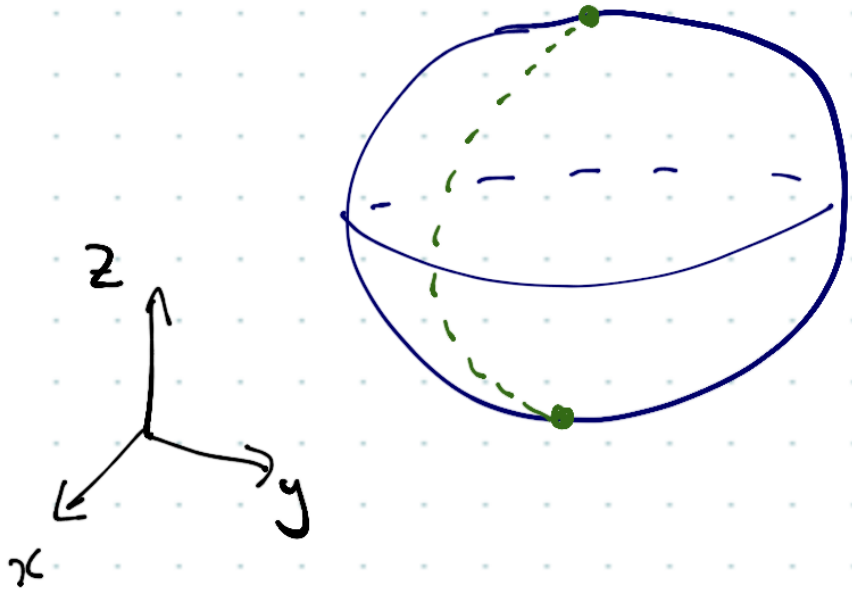
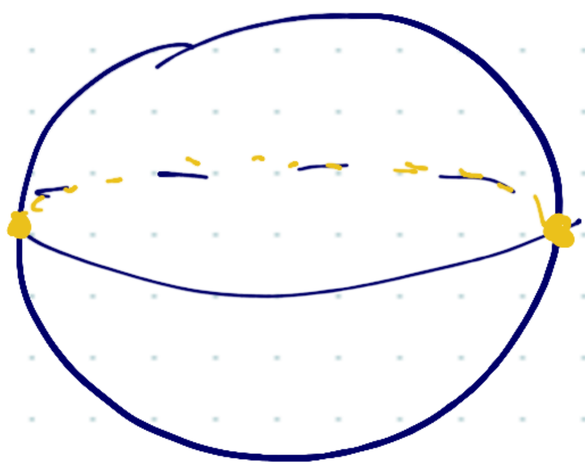
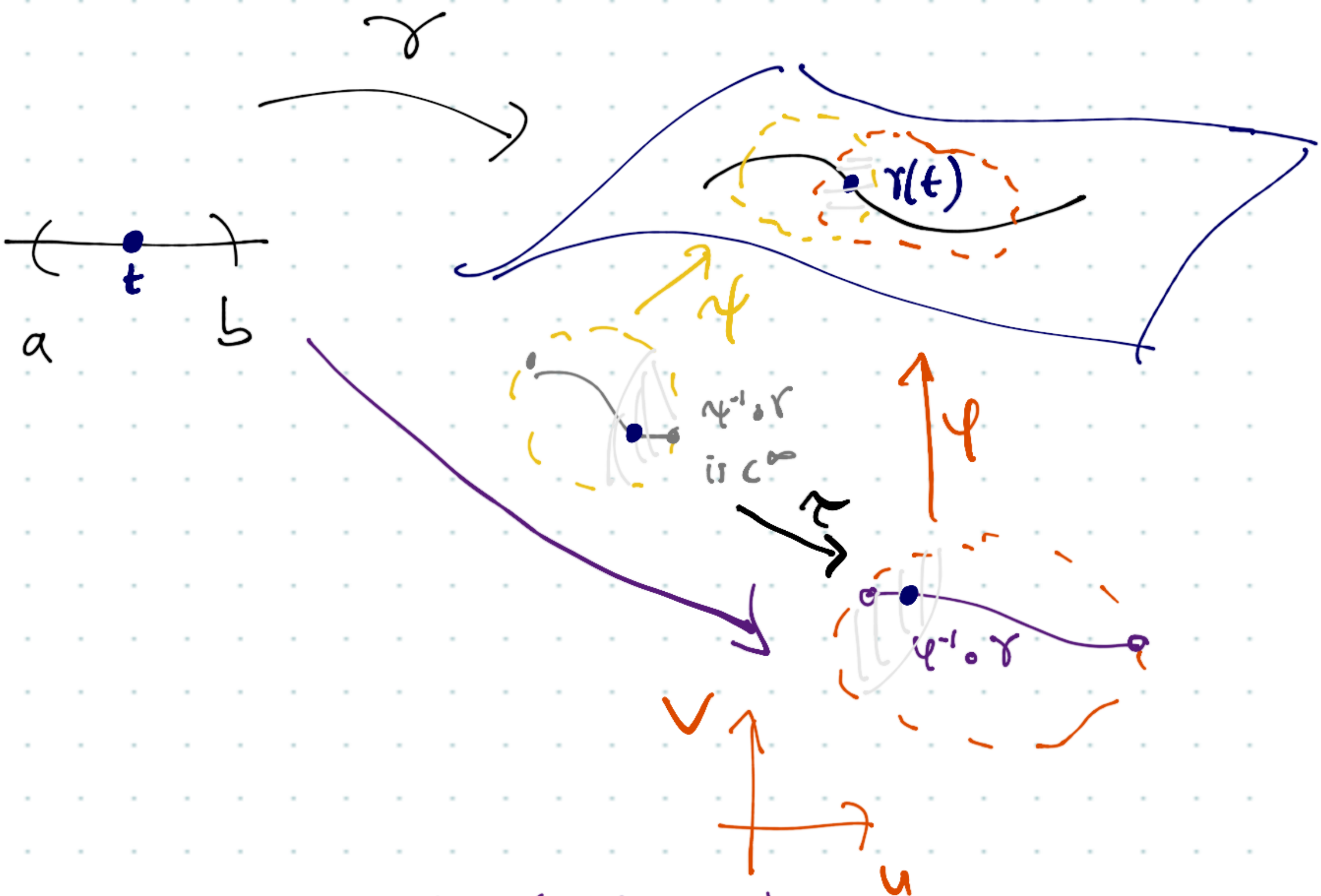

z
y
x





$\varphi^{-1} \circ \gamma(t) = (u(t), v(t))$

is C^∞ if $u, v : (a,b) \to \mathbb{R}$

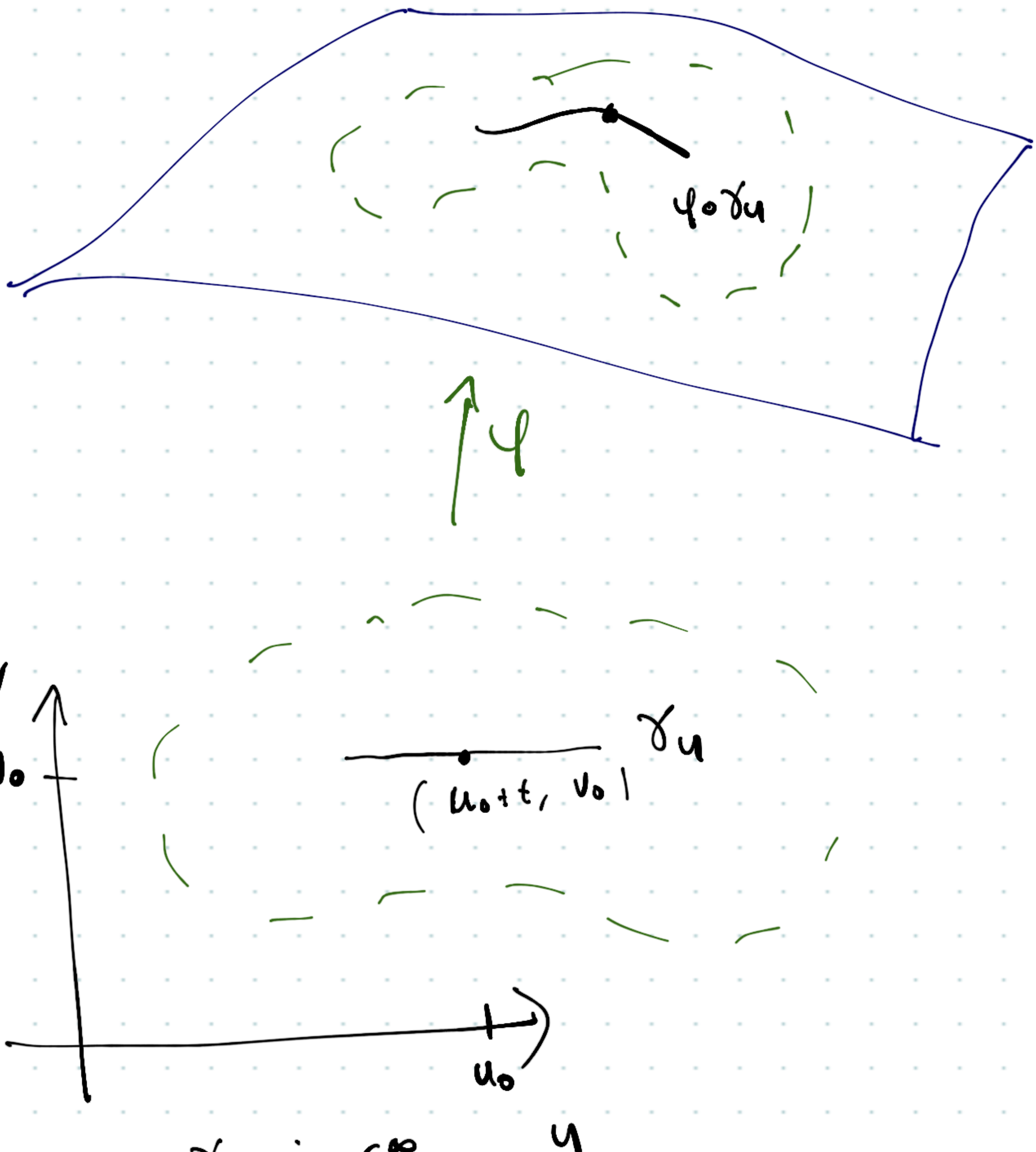
are C^∞.

$\tau = \varphi^{-1} \circ \psi$ is C^∞

if $\psi^{-1} \circ \gamma$ is C^∞

then $\varphi^{-1} \circ \gamma = \varphi^{-1} \circ \psi \circ \psi^{-1} \circ \gamma$

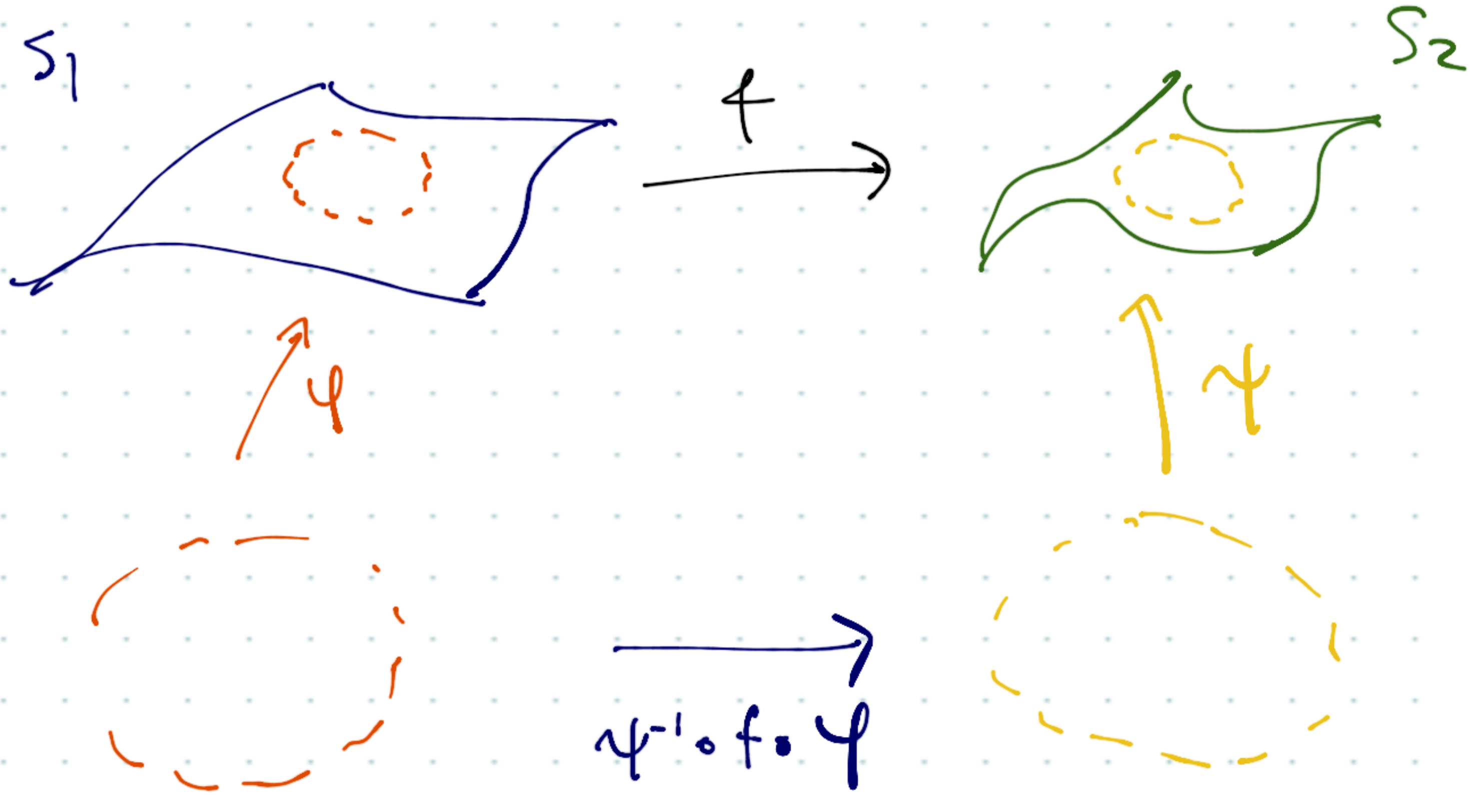
$= \tau \circ \psi^{-1} \circ \gamma$ is C^∞



Note γ_u is C^∞

$\therefore\ \varphi^{-1} \circ (\varphi \circ \gamma_u) = \gamma_u$ is C^∞

$\therefore\ \varphi \circ \gamma_u$ is C^∞ on S

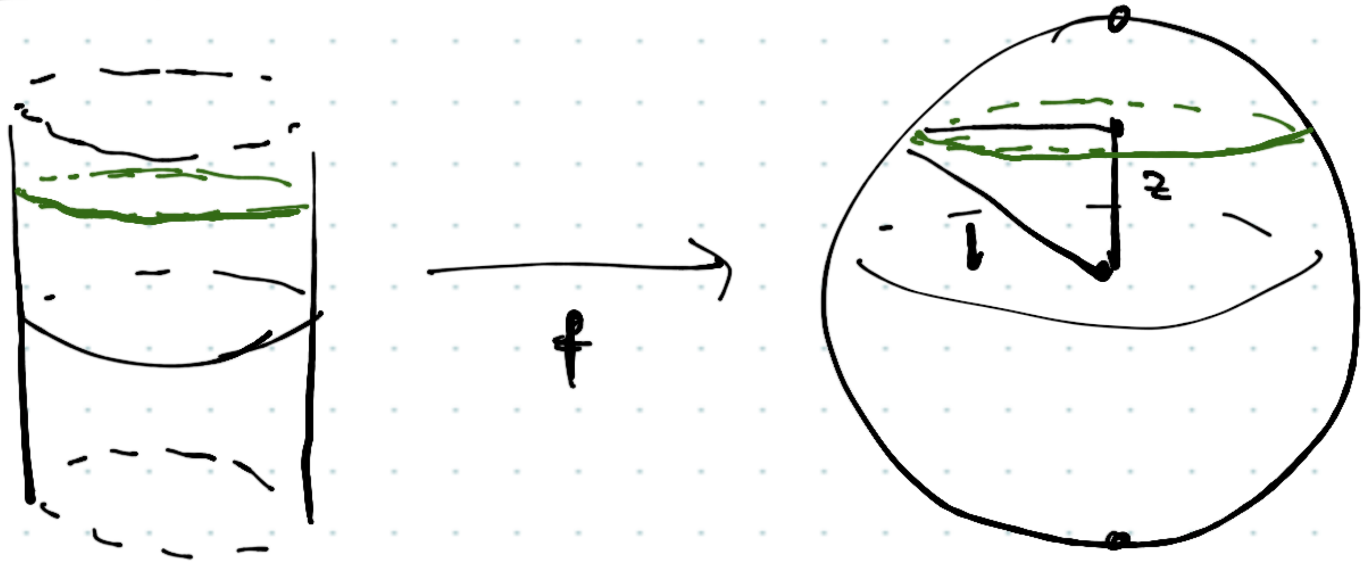


$\underbrace{f \text{ is } C^\infty}_{\text{defn}}$ if $\psi^{-1}\circ f\circ\varphi$ is C^∞

Eg:

$$(x,y,z) \longmapsto \left(\sqrt{1-z^2}\,x,\ \sqrt{1-z^2}\,y,\ z\right)$$

$r = \sqrt{1-z^2}$



Cylinder

$S^1 \times (-1,1)$

$\parallel$

$$\left\{ \begin{array}{l} x^2+y^2=1 \\ 0<z<1 \end{array} \right\}$$

Sphere

S^2

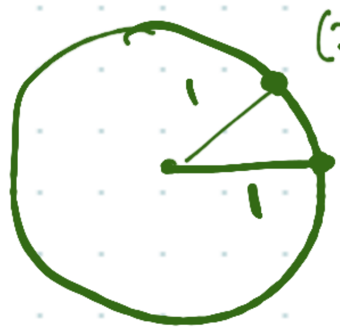
$\parallel$

$$\{x^2+y^2+z^2=1\}$$

$\left(\sqrt{1-z^2}\,x,\ \sqrt{1-z^2}\,y\right)$

$\parallel$

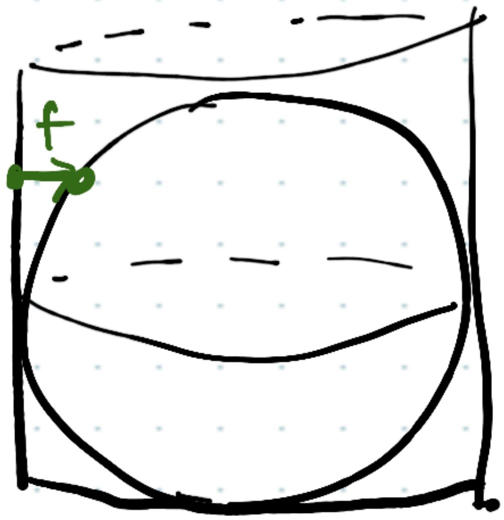
$(r\cos\theta, r\sin\theta)$



$(x,y) = (\cos\theta, \sin\theta)$

$r = \sqrt{1-z^2}$

Archimedes Tomb

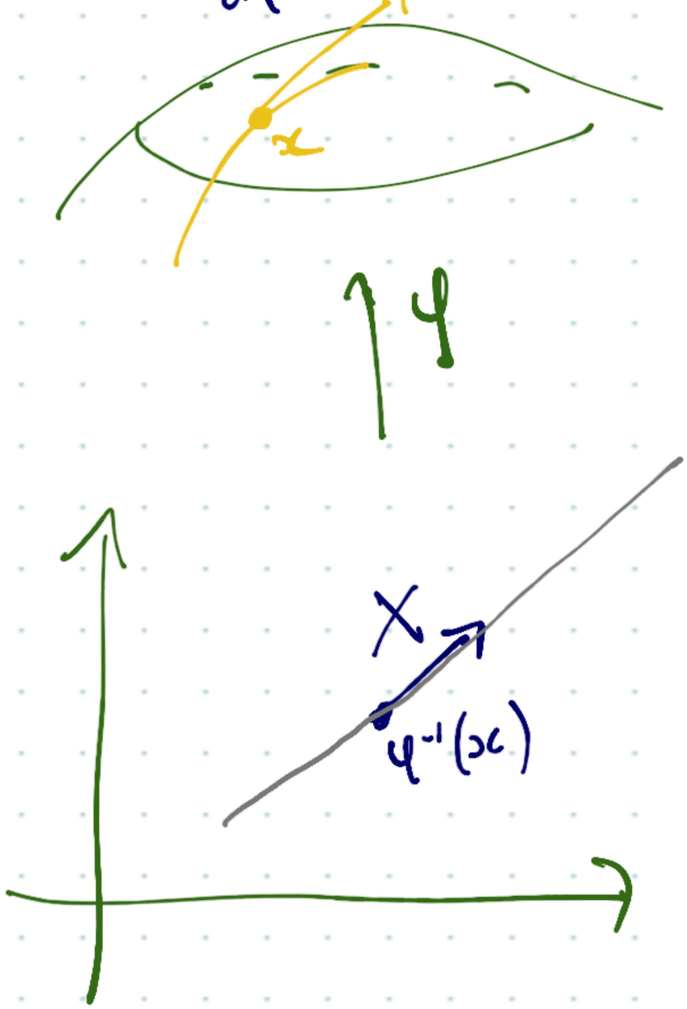


$d\varphi \cdot X = \gamma'(0) \in T_x S$

$$\gamma'(0) = \frac{d}{dt}\Big|_{t=0} \varphi \circ \varphi^{-1} \circ \gamma$$

$$= d\varphi_y \cdot \underbrace{(\varphi^{-1} \circ \gamma)'(0)}_{X}$$

where $\varphi(y) = x$



$\varphi^{-1}(x) + tX$

$$T_x S = \{ \gamma'(0) : \gamma(0) = x \}$$

$$T_x S = \{ d\varphi_y \cdot X : \varphi(y) = x \}$$

Notice γ is a C^∞ curve on S
iff $\varphi^{-1} \circ \gamma$ is a C^∞ in $\mathbb{R}^2$

$$T_x S = d\varphi_y(\mathbb{R}^2)$$

$$= \{ d\varphi_y \cdot X : X \in \mathbb{R}^2 \}$$

$$\subseteq \mathbb{R}^3$$ is a linear subspace

Note that if $\mathbb{R}^2 = \operatorname{span}\{e_u, e_v\}$

$e_u = (1,0)$

$e_v = (0,1)$

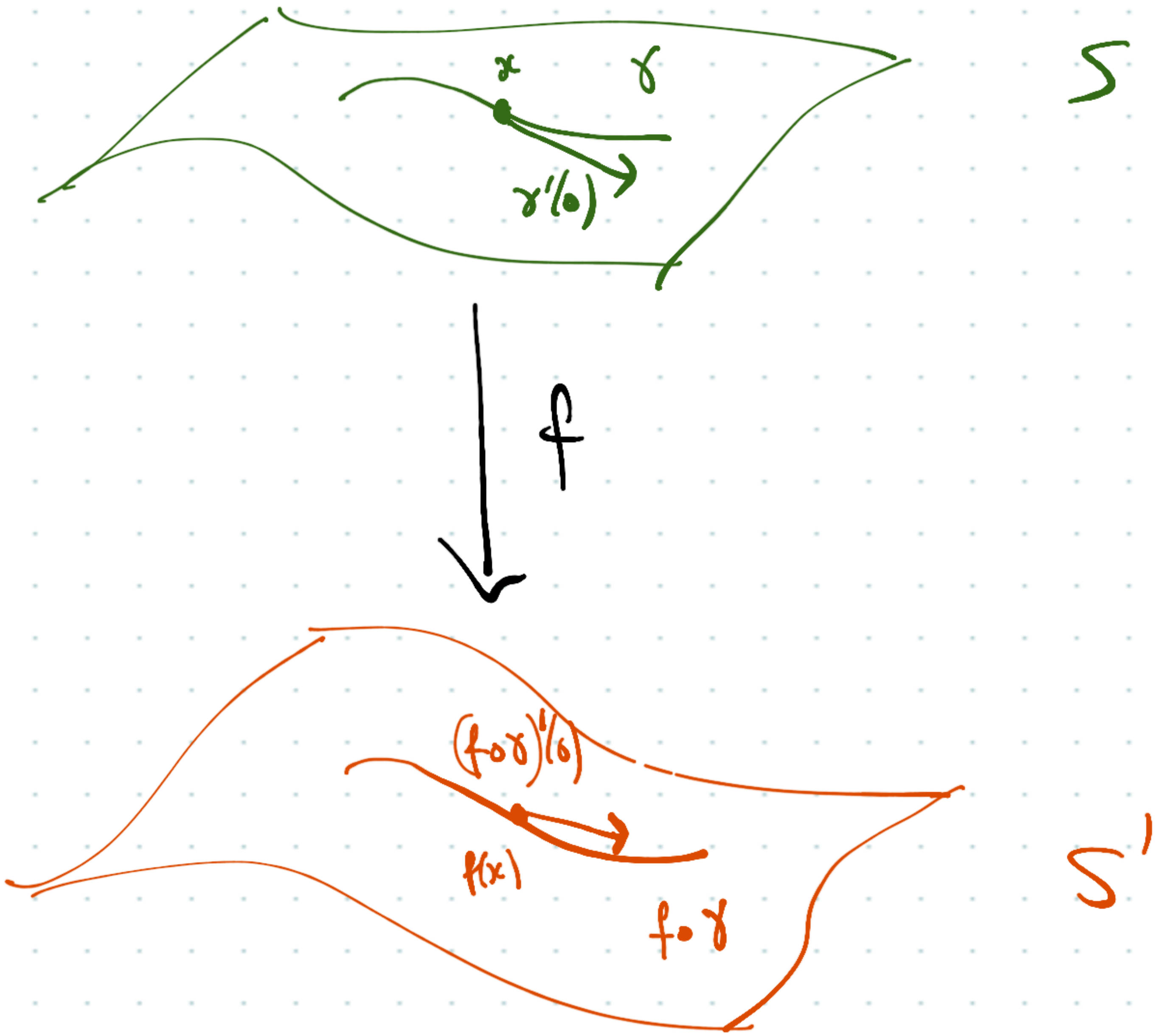
then $\{d\varphi \cdot e_u, d\varphi \cdot e_v\}$ are lin. indp.

hence span a plane.

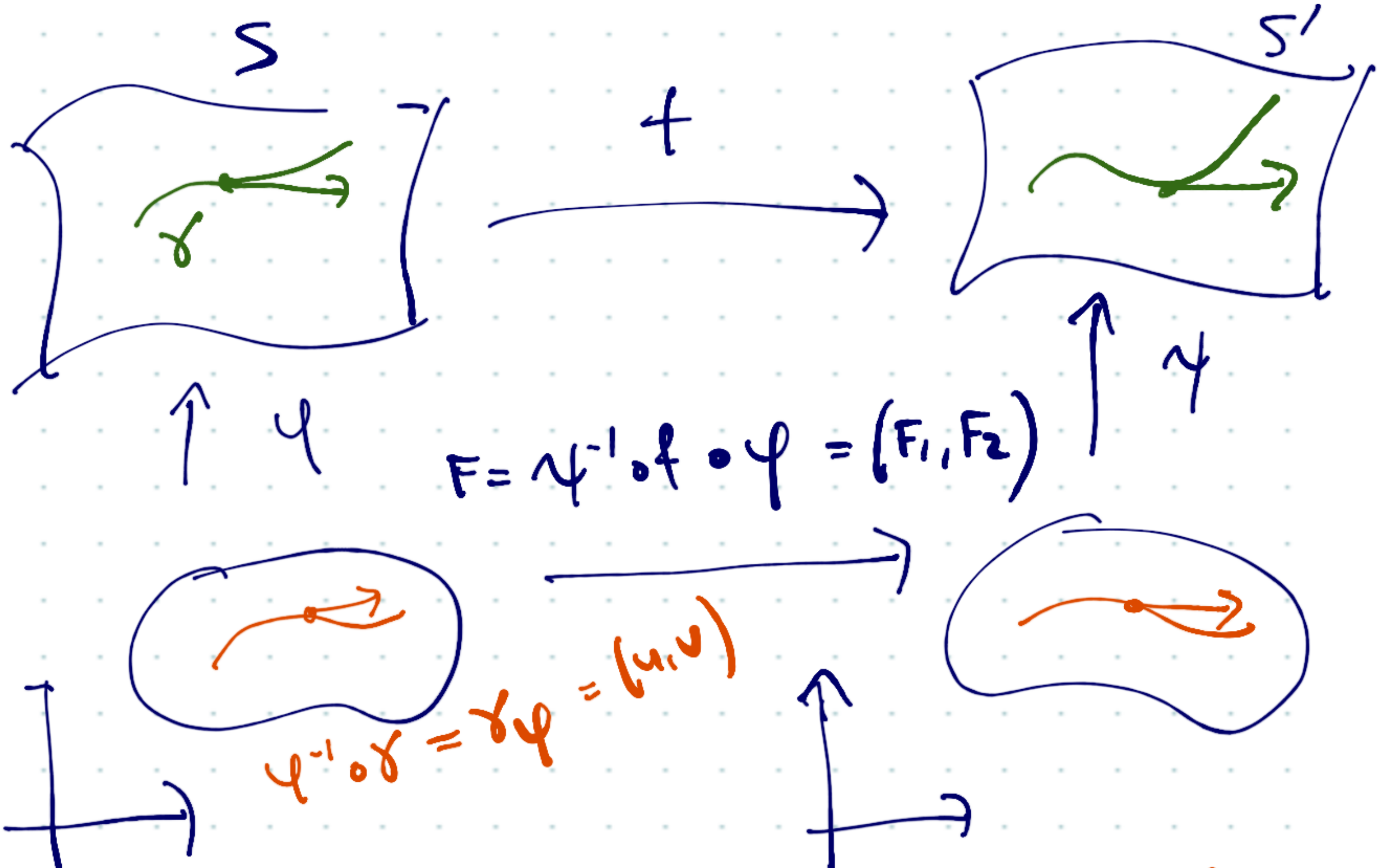
$d\varphi$ inj $\Rightarrow$

$$0 = c_1 d\varphi \cdot e_u + c_2 d\varphi \cdot e_v = d\varphi(c_1 e_u + c_2 e_v)$$

$\Rightarrow c_1 = c_2 = 0.$ since $d\varphi$ inj.



x
γ
γ'(0)
S
f
(f∘γ)'(0)
f(x)
f∘γ
S'



$$\psi^{-1}\circ f\circ\varphi\circ\varphi^{-1}\circ\gamma = F\circ\varphi^{-1}\circ\gamma$$

$$\psi^{-1}\circ f\circ\gamma$$

$$dF\cdot\gamma_\varphi'(0) = \frac{d}{dt}\Big|_{t=0} F(\gamma_\varphi(t))$$

$$= \frac{d}{dt}\Big|_{t=0}\left(F_1(u(t),v(t)),\ F_2(u(t),v(t))\right)$$

$$= \left(\frac{\partial F_1}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F_1}{\partial v}\frac{\partial v}{\partial t},\ \frac{\partial F_2}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F_2}{\partial v}\frac{\partial v}{\partial t}\right)$$

evaluated at $t=0$

$$dF \cdot \gamma_\varphi'(0) = \underbrace{\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}}_{dF.} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$$

Check that

$$d\psi \cdot \left(dF \cdot \gamma_\varphi'(0) \right) = df \cdot \gamma'(0)$$

use $\gamma_\varphi = \varphi^{-1} \circ \gamma$

$$F = \psi^{-1} \circ f \circ \varphi$$

use chain rule

$$dF = d\psi^{-1} \cdot df \cdot d\varphi$$

$$d\varphi = \begin{pmatrix} \partial_u \varphi & \partial_v \varphi \end{pmatrix} = 3\times 2 \text{ matrix}$$

$$= \begin{pmatrix} \partial_{x^1}\varphi & \partial_{x^2}\varphi \end{pmatrix}$$

$$d\varphi(\underbrace{c_1 e_1 + c_2 e_2}_{\mathbb{R}^2}) = c_1 \partial_{x^1}\varphi + c_2 \partial_{x^2}\varphi$$

$$T_x S = \{ d\varphi(c_1 e_1 + c_2 e_2) : (c_1, c_2) \in \mathbb{R} \}$$

$$c_1 d_1 g_{11} + c_2 d_2 g_{22} + c_1 d_2 g_{21} + c_2 d_1 g_{12}$$

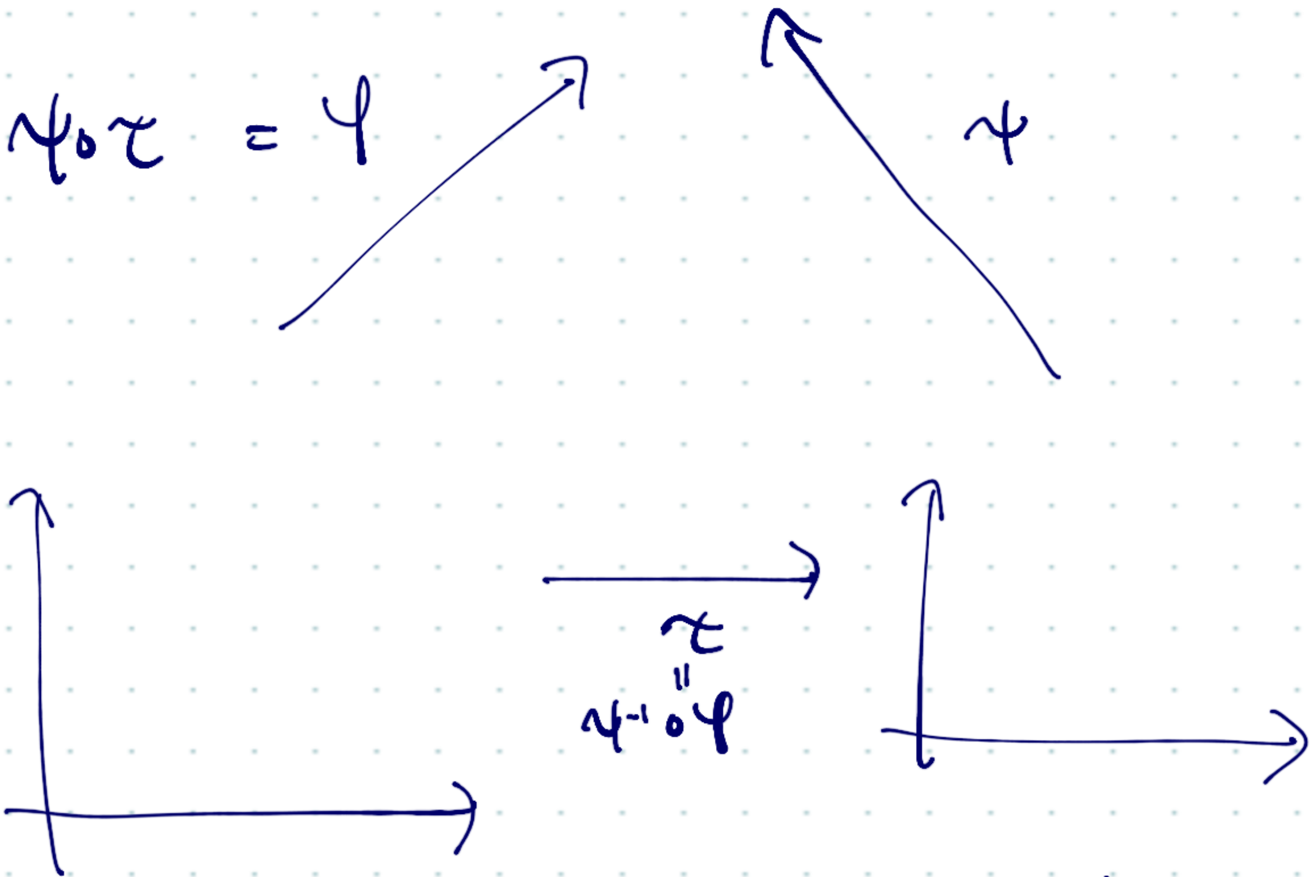
$$\parallel$$

$$\begin{pmatrix} d_1 & d_2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\parallel$$

$$\langle d^T, g \cdot c \rangle_{\mathbb{R}^2}$$

$$\psi\circ\tau = \varphi$$



$$X_\varphi \longmapsto d\tau\cdot X_\varphi$$

$$g^\varphi(X_\varphi, X_\varphi) = g^\psi(d\tau\cdot X_\varphi,\, d\tau\cdot X_\varphi)$$

$$g^\varphi(X_\varphi, X_\varphi) = g^{\psi\circ\tau}(X_\varphi, X_\varphi) = \langle d\varphi\cdot X_\varphi,\, d\varphi\cdot X_\varphi\rangle$$

$$g^\psi(d\tau\cdot X_\varphi,\, d\tau\cdot X_\varphi) = \langle d\psi\cdot d\tau\cdot X_\varphi,\, d\psi\, d\tau\cdot X_\varphi\rangle = \langle d\varphi\cdot X_\varphi,\, d\varphi\cdot X_\varphi\rangle$$