# MATH704 Differential Geometry

Macquarie University, Semester 2 2018

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# Lecture One: Introduction

# Lecture One: Introduction - Course Outline

# (Rough) Lecture Schedule

- Introduction: Week 1.
- Curves: Week 2.
- Surfaces: Weeks 3-5.
- Intrinsic Geometry (Riemannian Manifolds): Weeks 6-8.
- Curvature: Weeks 9-11.
- Global Geometry: Week 12.

# Assessment

- 3 Assignments (roughly equally spaced) x 15% = 45 %
- Final exam = 55%

# Books and Lecture Notes

- Lecture Notes
  - Differential Geometry
    http://pabryan.github.io/pdf/teaching/dg/dg.pdf
  - Lectures on Differential Geometry by Ben Andrews
    http://maths-people.anu.edu.au/~andrews/DG/
- Curves and Surfaces
  - do Carmo: Differential geometry of curves and surfaces
  - Montiel and Ros: Curves and surfaces
- Differentiable Manifolds
  - Lee: Introduction to Smooth Manifolds
  - Hitchin: Differentiable Manifolds
    http://people.maths.ox.ac.uk/hitchin/files/LectureNotes/Differentiable_manifolds/manifolds2014.pdf

# Riemannian Geometry (further reading)

- Do Carmo: Riemannian Geometry (a classic text that is certainly relevant today but sometimes considered a little terse. Does include material on differentiable manifolds.)
- Lee: Riemannian Manifolds: An Introduction to Curvature (very readable)
- Chavel: Riemannian Geometry: A Modern Introduction (more advanced, extensive discussion of many aspects of Riemannian Geometry)
- Petersen: Riemannian Geometry (more advanced, slightly non-standard approach definitely worth a look at some point)
- Gallot, Hulin, Lafontaine: Riemannian Geometry (more advanced, but very nice development of the formalism of Riemannian Geometry)

# Other Resources

- Discussion groups?
  - `http://slack.com/`
  - `http://piazza.com/`
  - others?
- Computational Techniques/Exploration
  - `https://cocalc.com/`
  - I produce all figures there and do some calculations also
  - See in particular `https://sagemanifolds.obspm.fr/`
  - Options for *computationally focused* assessment
- Many possible future research projects: undergraduate research project, honours, masters, Ph.D.,. . .

# Lecture One: Introduction - What is Differential Geometry?

# Calculus in Euclidean Space

Let $f : \mathbb{R} \to \mathbb{R}$.

$$\partial_x f = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note: To differentiate we need a linear structure on the *domain* to get $x + h$.

Let $f : \mathbb{R}^n \to \mathbb{R}$, $X \in \mathbb{R}^n$.

$$df_x \cdot X = \partial_t|_{t=0} f(\gamma(t))$$

where $\gamma(0) = x$ and $\gamma'(0) = X$. For example, $\gamma(t) = x + tX$.

Note: We need a linear structure on $\mathbb{R}^n$ to define

$$\gamma'(0) = \lim_{h \to 0} \frac{\gamma(h) - \gamma(0)}{h}.$$

# Curvilinear Calculus

- Unit sphere: $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$
- Let $f : \mathbb{S}^2 \to \mathbb{R}$ be a function. *How do we differentiate it?*

Consider the curve

$$\gamma(t) = (\cos(t), \sin(t), 0) \in \mathbb{S}^2.$$

We would like that

$$df_{(1,0,0)}(0,1,0) = \partial_t|_{t=0} f(\gamma(t)).$$

If $f = \bar{f}|_{\mathbb{S}^2}$ for $\bar{f} : \mathbb{R}^3 \to \mathbb{R}$, then

$$df_{(1,0,0)}(0,1,0) := \partial_t|_{t=0} \bar{f}(\gamma(t)).$$

- Does the result depend on $\bar{f}$?
- Does the result depnd on $\gamma$?
- Do we have to use the "ambient" $\mathbb{R}^3$ structure?

# Coordinates

Parametrise part of $\mathbb{S}^2$ by

$$\varphi(u, v) = (\cos u \sin v, \sin u \sin v, \cos v), \quad 0 < u < 2\pi, \quad 0 < v < \pi.$$

Then let

$$\tilde{f}(u, v) := f \circ \varphi(u, v) = f(\cos u \sin v, \sin u \sin v, \cos v)$$

- Now $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ and we can use Euclidean calculus!
- But does the result depend on $\varphi$?

## Example

Let $f = \bar{f}|_{\mathbb{S}^2}$ be defined by

$$\bar{f}(x, y, z) = xy + z$$

Let $\gamma(t) = (\cos(t), \sin(t), 0) \in \mathbb{S}^2$.

$$df_{(1,0,0)}(0, 1, 0) = \partial_t|_{t=0}\bar{f}(\gamma(t)) = \partial_t|_{t=0}[\cos(t)\sin(t) + 0] = 1.$$

On the other hand: $\gamma(t) = \varphi(t, \pi/2)$ and

$$\tilde{f}(u, v) = \underbrace{\cos(u)\sin(v)}_{x} \cdot \underbrace{\sin(u)\sin(v)}_{y} + \underbrace{\cos(v)}_{z}$$

Then

$$d\tilde{f}_{(0,\pi/2)}(1, 0) = \partial_t|_{t=0}[\cos(t)\sin(\pi/2) \cdot \sin(t)\sin(\pi/2) + \cos(\pi/2)] = 1.$$

# Geometry of curved surfaces

- Let $\varphi : \mathbb{R}^2 \to \mathbb{S}^2 \subseteq \mathbb{R}^3$ as before.
- Coordinate curves. Fix, $u_0, v_0$:

$$\gamma_{u_0}(t) = \varphi(u_0, t), \quad \gamma_{v_0}(t) = \varphi(t, v_0).$$

- Coordinate vectors

$$e_u = \partial_u \varphi, \quad e_v = \partial_v \varphi.$$

- Angle and length:

$$|e_u| = \sqrt{\langle e_u, e_u \rangle}, \quad |e_v| = \sqrt{\langle e_v, e_v \rangle}, \quad \cos\theta = \frac{\langle e_u, e_v \rangle}{|e_u||e_v|}.$$

Note: These are *functions* of $u, v$!

# Example

$$\varphi(u,v) = (\cos u \sin v, \sin u \sin v, \cos v), \quad 0 < u < 2\pi, \quad 0 < v < \pi.$$

$$e_u = (-\sin u \sin v, \cos u \sin v, 0), \quad e_v = (\cos u \cos v, \sin u \cos v, -\sin v)$$

$$|e_u| = \sqrt{(-\sin u \sin v)^2 + (\cos u \sin v)^2} = |\sin v| = \sin v.$$

$$|e_v| = \sqrt{(\cos u \cos v)^2 + (\sin u \cos v)^2 + (-\sin v)^2} = 1.$$

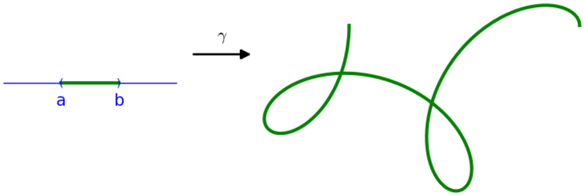
$\langle e_u, e_v \rangle = 0 \Rightarrow \theta = \pi/2$. Check This!

# Lecture One: Introduction - Curves and Surfaces

# What is a curve?

## Definition

A *parametrised curve* in the plane is a smooth function $\gamma : (a, b) \to \mathbb{R}^2$. In addition, $\gamma$ is *regular* if $\gamma'(t) \neq 0$ for all $t \in (a, b)$.



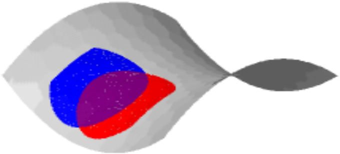

- Regularity is *very* important. It allows us to transfer calculus on $(a, b)$ to calculus on Image $\gamma := \{\gamma(t) : t \in (a, b)\} \subset \mathbb{R}^2$.
- *Space curves* are the same but in $\mathbb{R}^3$.

# What is a surface?

## Definition

A *regular surface* $S \subseteq \mathbb{R}^3$ is a subset of $\mathbb{R}^3$ such that there are *local parametrisations* $\varphi_i : U_i \underset{\text{open}}{\subseteq} \mathbb{R}^2 \to \mathbb{R}^3$ that are smooth maps with

1. $S = \cup_i V_i$ where $V_i = \varphi_i(U_i)$,
2. $\varphi_i$ is a homeomorphism onto it's image $V_i = \varphi_i(U_i)$
3. $d\varphi_i|_x : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for each $x \in U_i$.



# What is a surface

Writing $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$:

$$d\varphi = \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{pmatrix}$$

- Injectivity means the tangent plane

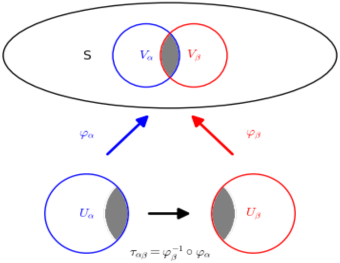
  $$T_p S = \text{span}\{d\varphi_{(u,v)}(1,0), \quad d\varphi_{(u,v)}(0,1)\} = \text{Image } d\varphi_{(u,v)}.$$

  exists at the point $p = \varphi(u,v)$.
- Injectivity also allows us to transfer calculus from open sets $U_i$ in the plane to $S$.

# Lecture One: Introduction - Intrinsic Geometry

# Manifolds

- Forget that $S$ is a subset of $\mathbb{R}^3$ and find some *intrinsic structure*.
- The key idea is that we only need the local parametrisations and *compatability*
- For each $i, j$, the map $\tau_{ij} = \phi_i^{-1} \circ \phi_j$ is a *diffeomorphism*. That is, differentiable with differentiable inverse.



# Geometry of Riemannian metric

Gauss' big idea!

- Define the (Riemannian) metric tensor

$$g = \begin{pmatrix} \langle \partial_u \varphi, \partial_u \varphi \rangle & \langle \partial_u \varphi, \partial_v \varphi \rangle \\ \langle \partial_v \varphi, \partial_u \varphi \rangle & \langle \partial_v \varphi, \partial_v \varphi \rangle \end{pmatrix}$$

- Length and angle determined by the inner-product $g$:

$$g(X, Y) = (X^u, X^v) \cdot g \cdot (Y^u, Y^v)^T$$

  where $X = X^u e_u + X^v e_v$ and similar for $Y$.

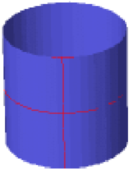
*Intrinsic Geometry* arises by forgetting that $g$ came from embedding into $\mathbb{R}^3$ and just thinking of it as a symmetric, positive definite matrix valued function!

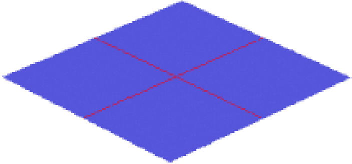
- Length, angle, and area are determined by $g$ alone.
- Gauss worked with surfaces and Riemann introduced the general notion of metric in $n$ dimensions: Riemannian metric, line element, first fundamental form.

# Intrinsic quantities

Gauss' Theorema Egregium (Remarkable Theorem):

- The *Gauss Curvature* is *intrinsic*. That is, it depends only on the geometry of the metric $g$ and not how the surface lies in space.





- The cylinder and plane have the same "flat" geometry.
- The plane has only straight "principal" lines.
- The cylinder has one straight and one circular principal line.
- The Gauss curvature is the product of the two *principal curvatures*.

# Lecture One: Introduction - Curvature

# Extrinsic Curvature

Geodesic curvature of curve:

$$\kappa = \langle \partial_s T, N \rangle = \langle \partial_s^2 \gamma, N \rangle.$$

- Second derivative with respect to arc-length.

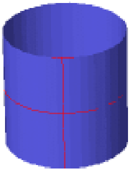
On a surface, the curvature of the *surface* (not the curve!) is the *normal* part of the curvature.

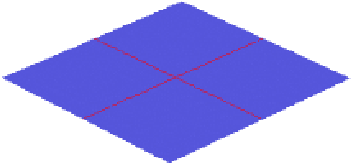
- That is, for $\gamma : (a, b) \to S$ a curve along $S$:

  $$\kappa_S(\gamma) = \langle \partial_s^2 \gamma, N_S \rangle$$

  where $N_S$ is the normal of the surface.

# Example





- Straight lines have zero curvature, while the unit circle has curvature 1.
- Plane has zero curvature, cylinder has zero curvature in one direction and curvature equal to 1 in another.

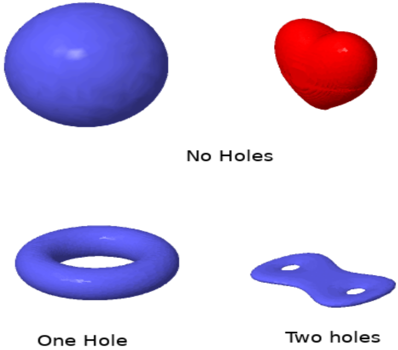
# Intrinsic Curvature

- *Gauss curvature*: $K = \kappa_1 \kappa_2$ where $\kappa_i$ are the *principal curvatures*.
  - Plane has $\kappa_1 = \kappa_2 = 0$.
  - Cylinder has $\kappa_1 = 0, \kappa_2 = 1$.
  - Sphere has $\kappa_1 = \kappa_2 = 1$.
- Gauss showed the Gauss curvature is intrinsic! Plane and cylinder have the same geometry but the sphere does not.
- *Mean Curvature*: $H = \kappa_1 + \kappa_2$.
  - $H$ is *extrinsic* (i.e. not intrinsic). See plane and cylinder.
- Riemann introduced curvature tensor to measure intrinsic curvature.
  - It depends only on $g$ and not how the surface sits in space.
  - However, the intrinsic curvature of $S$ and the extrinsic curvature of $S$ are related by the *Gauss equation*.
- The Einstein-Hilbert equations in General Relativity are equations for the *intrinsic curvature*.
  - After all, this is the curvature of space-time itself and not how space-time sits in some larger ambient space!

# Lecture One: Introduction - Global Geometry

# Classification of Closed Surfaces

- Closed surfaces are classified by genus $\lambda \in \mathbb{N}$ (number of holes)
- Topological invariant



# Gauss-Bonnet

$$\int_S K d\sigma = 4\pi(1 - \lambda)$$

The left hand side is geometric while the right hand side is topological!

- A sphere has $\int_S K d\sigma = 4\pi > 0$: "more" positive curvature on average.
- A torus has balanced positive and negative curvature: either it's flat or it has points of positive and points of negative curvature
- Higher genus surfaces: "more" negative curvature on average

# Constant Sectional Curvature

- Constant sectional curvature manifolds have constant curvature!
- There are three main cases:
  - $K > 0$: Sphere
  - $K = 0$: Euclidean Space
  - $K < 0$: Hyperbolic Space
- These models are all *simply connected* (no holes)
- In general, there may be more complicated topology
  - But then constant curvature implies quotient of one of the three main cases
- These are called *spaceforms*