MATH704 Differential Geometry

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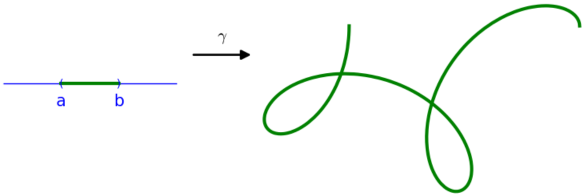
Lecture Two: Curves

Lecture Two: Curves - Parametrised Curves

Regular Parametrised Curves

Definition

A smooth *parametrised curve* in the plane is a smooth function $\gamma : (a, b) \to \mathbb{R}^2$. In addition, γ is *regular* if $\gamma'(t) \neq 0$ for all $t \in (a, b)$.



- Regularity is *very* important. It allows us to transfer calculus on (a, b) to calculus on Image $\gamma := \{\gamma(t) : t \in (a, b)\} \subset \mathbb{R}^2$.
- *Space curves* are the same but in $\mathbb{R}^3$.

Examples of Curves

Example

$$\gamma_1(t) = (\cos(t), \sin(t)), -\pi < t < \pi.$$

$$\gamma_2(t) = (\cos(t^2), \sin(t^2), -\sqrt{\pi} < t < \sqrt{\pi}.$$

Notice that $\mathrm{Img}(\gamma_1) := \{\gamma_1(t) : -\pi < t < \pi\} = \mathrm{Img}(\gamma_2)$ but $\gamma_1 \neq \gamma_2$. The first is regular, but $\gamma_2'(0) = 0$ so γ_2 is not regular.

Example

$$\gamma(t) = (t, |t|), t \in \mathbb{R}.$$

This time γ is not differentiable at $t = 0$ so is not even a smooth parametrised curve.

Examples of Curves

Example

$$\gamma(t) = (t^3, t^2), \quad t \in \mathbb{R}.$$

We have $\mathrm{Img}(\gamma) = \{y = x^{2/3}\}$ has a *cusp* at $t = 0$. This time, there is no regular parametrisation of $\mathrm{Img}(\gamma)$! *See the implicit function theorem.*

Example

$$\gamma(t) = (t^3 - 4t, t^2 - 4).$$

Here γ is regular, but it is not one-to-one. That is, it crosses itself.

Example

$$\gamma(t) = (\cos(t), \sin(t)).$$

Here γ is one-to-one on $(0, 2\pi)$ but not on any larger interval. However, $\gamma^{(k)}(0) = \gamma^{(k)}(2\pi)$ so that γ smoothly *closes up* to give a closed curve. Any smooth periodic function satisfies this property.

Lecture Two: Curves - Change of Parameters

Change of Parameters

- Let $\gamma : (a, b) \to \mathbb{R}^2$ and $\sigma : (c, d) \to \mathbb{R}^2$ be regular parametrisations with $C := \operatorname{Img} \gamma = \operatorname{Img} \sigma$.
- Assume for the moment that γ and σ are one-to-one so that $\gamma^{-1} : C \to (a, b)$ is defined and $\sigma^{-1} : C \to (c, d)$ is defined.
- We call $\varphi = \sigma^{-1} \circ \gamma : (a, b) \to (c, d)$ the *change of parameters*.

Lemma

The function φ is a diffeomorphism. That is, it is a smooth function with smooth inverse.

Differential of Change of Parameters

Proof.

- Let $T = \gamma' / |\gamma'|$ be the unit tangent (regularity!) and $N = J(T)$ be the unit normal with J rotation by $\pi/2$.
- Define the function

$$\Gamma(t, u) = \gamma(t) + uN(t).$$

- The differential is the matrix

$$d\Gamma = \begin{pmatrix} \gamma' + uN' & N \end{pmatrix}$$

  Note here we have two columns!

- Now observe that $\gamma(t) = \Gamma(t, u = 0)$ and the differential is non-singular:

$$d\Gamma(t, u = 0) = \begin{pmatrix} |\gamma'| \, T & N \end{pmatrix}$$

Proof of Change of Parameters Lemma

Proof.

- By the *inverse function theorem* (see next lecture!), for each t, there is an open set U containing $(t, 0)$ and an open set V in $\mathbb{R}^2$ containing $\gamma(t)$ such that

$$\Gamma|_U : U \underset{\simeq}{\rightarrow} V$$

  is a diffeomorphism with $\gamma(t) = \Gamma(t, u = 0)$.

- Likewise applying the same argument to σ we have

$$\Sigma|W : W \rightarrow Z$$

  is a diffeomorphism with $\sigma(s) = \Sigma(s, v = 0)$.

- But now

$$\sigma^{-1} \circ \gamma = \Sigma^{-1}|_C \circ \Gamma|_{u=0}$$

  is differentiable with differentiable inverse $\Gamma^{-1}|_C \circ \Sigma|_{s=0}$.

□

Straightening

- The use of the map Γ is known as *straightening* the curve γ because it identifies an open set around a point of C with an open set of $\mathbb{R}^2$ such that C is identified with the horizontal axis.

Inverse Function Theorem

- The key ingredient was the inverse function theorem. We will investigate this more closely next week.

For now, as an illustration, note that if $f : \mathbb{R} \to \mathbb{R}$ satisfies $f'(x_0) \neq 0$, then f is monotone on an interval $(x_0 - \epsilon, x_0 + \epsilon)$ and hence invertible on that interval. Moreover, the inverse is differentiable. This is precisely the 1-dimensional inverse function theorem.

Notice that if $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ is a diffeomorphism, then since $f \circ f^{-1} = \mathrm{Id}$, by the chain rule

$$df \circ df^{-1} = d(f \circ f^{-1}) = d\,\mathrm{Id} = \mathrm{Id}$$

and hence df is non-singular. The inverse function theorem is a *local* converse.

For example, the function $f(x) = x^3$ has $f'(0) = 0$ so *cannot* be smoothly invertible near $x = 0$. However, f^{-1} does exist: $f^{-1}(y) = y^{1/3}$ which is not differentiable at $y = f(0) = 0$.

Lecture Two: Curves - Arc Length

Arc Length Parameter

Definition

The arc length parameter $s(t) = \int_a^t |\gamma'(\tau)|\, d\tau$.

Lemma

The arc length parameter $s(t)$ is smoothly invertible so we may write $t = t(s)$. Then the parametrisation $\bar{\gamma}(s) = \gamma(t(s))$ satisfies $|\bar{\gamma}'| \equiv 1$.

Proof.

1. $s'(t) = |\gamma'(t)| \neq 0$ hence s is smoothly invertible.
2. $$\partial_s \bar{\gamma}(s) = \gamma'(t(s))\partial_s t(s) = \gamma'(t(s))\frac{1}{s'(t(s))} = \frac{\gamma'(t(s))}{|\gamma'(t(s))|}.$$

   Therefore $\partial_s \bar{\gamma}$ is unit length as required.

□

Arc Length of Curves

Definition

For $p, q \in [a, b]$, the *arc length* along γ between $\gamma(p)$ and $\gamma(q)$ is

$$\ell(p, q) = \int_p^q |\gamma'| \, dt.$$

The total length of γ is

$$L(\gamma) = \int_a^b |\gamma'| \, dt = s(b).$$

In the arc length parametrisation $s \in (0, L(\gamma))$,

$$\ell(s_1, s_2) = \int_{s_1}^{s_2} |\gamma'| \, ds = \int_{s_1}^{s_2} ds = |s_2 - s_1| \, .$$

Exercise!: Invariance under change of parameters, $\varphi : (a, b) \to (c, d)$:

$$\int_c^d |\gamma'(t)| \, dt = \int_a^b |(\gamma \circ \varphi)'(u)| \, du$$

Length as a Riemann Sum

- Let $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ be a partition of $[a, b]$.
- Then for N large, so that for example $t_{i+1} - t_i = \Delta t := (b-a)/N$ is small
$$l(t_i, t_{i+1}) \simeq \left|\gamma'(t_i)\right| \Delta t$$
- Then
$$\int_a^b \left|\gamma'(t)\right| dt = \lim_{N\to\infty} \sum_{i=1}^N \left|\gamma'(t_i)\right| \Delta t.$$
- That is, the arc length of γ is obtained by approximating γ by short straight lines and adding up their lengths.

Polygonal Approximation

- Exercise: Let $L_i = |\gamma(t_{i+1}) - \gamma(t_i)|$ be the length of the line segment joining $\gamma(t_{i+1})$ to $\gamma(t_i)$. Then

$$\int_a^b |\gamma'| \, dt = \lim_{N \to \infty} \sum_{i=1}^{N} L_i.$$

- Challenge Exercise

$$\int_a^b |\gamma'| \, dt = \sup \sum_i L_i$$

where the supremum is taken over all partitions of $[a, b]$.

Lecture Two: Curves - Curvature

Geodesic Curvature

- Parametrise by arc length
- Unit tangent: $T = \gamma'$ (regularity!)
- Unit normal: $N = J(T)$ where J is rotation by $\pi/2$.
  - Either clockwise or counter-clockwise is fine giving $\langle T, N\rangle = 0$. But we *must* make a single consistent choice to ensure N is continuous.

Since $\langle T, T\rangle \equiv 1$ we have

$$0 = \partial_s \langle T, T\rangle = 2\langle \partial_s T, T\rangle.$$

That is

$$\partial_s T = \kappa N$$

for some function κ.

Definition

The *geodesic curvature* (with respect to N) of γ is $\kappa = \langle \partial_s T, N\rangle$.

Frenet-Serret Frame

- For each point $\gamma(t)$, $\{T(t), N(t)\}$ is an orthonormal basis for $\mathbb{R}^2$.
- We think of $T(t), N(t)$ as vectors based at $\gamma(t)$. As t varies, the base point varies. For this reason, $\{T(t), N(t)\}$ is known as a *Moving Frame*.
  - For curves, this frame is also called the *Frenet-Serret Frame*.

Lemma (Frenet-Serret Equations)

$$\partial_s \begin{pmatrix} T \\ N \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$$

- Exercise: Differentiate $\langle T, N \rangle = 0$ to prove the lemma.

Change of Ambient Orientation

Make the change $\bar{N}(s) = -N(s)$. This changes the orientation $T \to N$ of $\mathbb{R}^2$ to the orientation of $\mathbb{R}^2$ $T \to \bar{N} = -N$. That is, it swaps clockwise and counter-clockwise.

The curvature changes by

$$\bar{\kappa} = \langle \partial_s T, \bar{N} \rangle = -\langle \partial_s T, N \rangle = -\kappa.$$

The sign of the geodesic curvature is defined only up to a choice of orientation of $\mathbb{R}^2$

Change of Curve Orientation

Suppose γ is parametrised by arc-length on (a, b). Reverse direction and parametrise by

$$\mu(s) = \gamma(-s), \quad s \in (-b, -a).$$

Then

$$T_\mu(s) = \mu'(s) = -\gamma'(-s) = -T_\gamma(-s).$$

and

$$N_\mu(s) = J(T_\mu(s)) = -J(T_\gamma(-s)) = -N_\gamma(s).$$

Then

$$\kappa_\mu(s) = \langle \partial_s T_\mu(s), N_\mu(s) \rangle = \langle \partial_s[-T_\gamma(-s)], -N_\gamma(-s) \rangle = -\kappa_\gamma(-s).$$

Reversing the orientation of γ (but not of $\mathbb{R}^2$) changes the sign of κ also!

Geometric Interpretation

- The curvature measures the *deviation* of γ from the tangent line $u \mapsto \gamma(s) + uT(s)$.
- Exercise: Show that $\kappa \equiv 0$ if and only if $\gamma(t) = p + tv$ is a straight line.
- Exercise: Show that $\kappa \equiv 1/r$ for some $r > 0$ if and only if $\gamma(s) = r(\cos(s + s_0), \sin(s + s_0)) + p$ is a circle of radius r centred on p.
  - *Hint*: It might be helpful to think about the next exercise first.
- Quadratic Approximation:

$$\begin{aligned}\gamma(s) &= \gamma(s_0) + (s - s_0)\gamma'(s_0) + \frac{1}{2}(s - s_0)^2\gamma''(s_0) + \cdots \\ &= \gamma(s_0) + (s - s_0)T(s_0) + \frac{1}{2}(s - s_0)^2\kappa(s_0)N(s_0) + \cdots\end{aligned}$$

The Curvature Determines the Curve

- Exercise: Show that given any smooth function κ, there exists a curve γ parametrised by arc-length with curvature κ. In fact, all such curves are of the form

$$\gamma(s) = \left(\int \cos\theta(s) ds, \int \sin\theta(s) \right) + p$$

where $p \in \mathbb{R}^2$ and

$$\theta(s) = \int \kappa(s) ds + \theta_0$$

with $\theta_0 \in \mathbb{R}$.

  - *Hint*: Use the fact that $T = \gamma'$ has unit length hence has the form $T = (\cos\theta(s), \sin\theta(s))$ for some smooth function θ (the implicit function theorem guarantees smoothness). Now determine N in terms of T and differentiate T to obtain an equation for θ in terms of κ. Then finally, integrate T to obtain γ.

- Exercise: Conclude that all arc-length parametrisations of the unit circle centered on the origin are of the form

$$\gamma(s) = (\cos(s + s_0), \sin(s + s_0)).$$

Invariance Under Rigid Motion

Definition

A *rigid motion* of the plane is any affine transformation

$$T(x) = A \cdot x + b, \quad x \in \mathbb{R}^2$$

$b \in \mathbb{R}^2$ and where A is an *orthogonal matrix*. That is

$$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^2.$$

- Exercise: Let γ be a regular parametrised curve and define a new regular parametrised curve $\mu(s) = T(\gamma(s)) = A \cdot \gamma(s) + b$.
  1. Show that $T_\mu = A \cdot T_\gamma$ and $N_\mu = \pm A \cdot N_\gamma$ (the sign depends on whether A preserves or reverses orientation).
  2. Show that if γ is parametrised by arc-length, then so is μ.
  3. Show that $\kappa_\mu(s) = \kappa_\gamma(s)$.

We say that κ is *invariant under rigid motion*.

Lecture Two: Curves - Space Curves

Normal and Binormal Vectors

A regular, parametrised *space curve* is a smooth map $\gamma : (a, b) \to \mathbb{R}^3$ with $\gamma' \neq 0$.
Unlike for plane curves, we cannot a-priori define a normal vector: Putting $T = \gamma' / |\gamma'|$,

$$T^{\perp}(t) = \{V \in \mathbb{R}^2 : \langle T(t), V \rangle = 0\}$$

is a two-dimensional plane passing through $\gamma(t)$.
As with plane curves however we may still parametrise by arc-length and then

$$1 \equiv \langle T, T \rangle \Rightarrow \gamma'' \perp T.$$

Therefore, if $\gamma'' \neq 0$, we may choose a unit normal vector N in $T^{\perp}$ and a *binormal vector* $B \in T^{\perp}$ to obtain an oriented basis $\{T, N, B\}$ of $\mathbb{R}^3$:

$$T(s) = \gamma', \quad N(s) = \frac{\gamma''}{|\gamma''|}, \quad B(s) = T(s) \times N(s).$$

Curvature and Torsion

We define the curvature,

$$\kappa(s) = \left|\gamma''(s)\right|.$$

For space curves, we will restrict to the curves with $\kappa > 0$ so that $N = \frac{\gamma'}{\kappa}$ is defined.

Here we cannot give a sign to the curvature since we cannot a-priori choose a normal vectors.

Since B is unit length, $\partial_s B \perp B$. Moreover

$$\partial_s B = \partial_s(T \times N) = T' \times N + T \times N' = T \times N' \perp T$$

since $T' = \kappa N \Rightarrow T' \times N = 0$.

Therefore we define the *torsion*, τ by

$$B' = -\tau N.$$

- Exercise: Since N is unit length, $\partial_s N \perp N$ and

$$\partial_s N = -\kappa T + \tau B$$

Frenet-Serret Frame

Now we have a three dimensional frame $\{T, N, B\}$.
The *Osculating Plane* is the plane spanned by T and N.
The Frenet-Serret equations are

$$\partial_s \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

- κ measures the deviation of γ from the tangent line in the osculating plane.
- τ measures the *twisting* of γ away from the osculating plane.
- A space curve γ with $\kappa > 0$ lies in a plane if and only if $\tau \equiv 0$.
- Given $\kappa > 0$ and τ, there exists a unique (up to rigid motion) curve with the given curvature and torsion.