MATH704 Differential Geometry

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Lecture Three: Curves

Lecture Three: Curves - Curvature

Geometric Interpretation

- The curvature measures the *deviation* of γ from the tangent line $u \mapsto \gamma(s) + uT(s)$.
- Exercise: Show that $\kappa \equiv 0$ if and only if $\gamma(t) = p + tv$ is a straight line.
- Exercise: Show that $\kappa \equiv 1/r$ for some $r > 0$ if and only if $\gamma(s) = r(\cos(s + s_0), \sin(s + s_0)) + p$ is a circle of radius r centred on p.
  - *Hint*: It might be helpful to think about the next exercise first.
- Quadratic Approximation:

$$\begin{aligned}\gamma(s) &= \gamma(s_0) + (s - s_0)\gamma'(s_0) + \frac{1}{2}(s - s_0)^2\gamma''(s_0) + \cdots \\ &= \gamma(s_0) + (s - s_0)T(s_0) + \frac{1}{2}(s - s_0)^2\kappa(s_0)N(s_0) + \cdots\end{aligned}$$

The Curvature Determines the Curve

- Exercise: Show that given any smooth function κ, there exists a curve γ parametrised by arc-length with curvature κ. In fact, all such curves are of the form

$$\gamma(s) = \left(\int \cos\theta(s) ds, \int \sin\theta(s) \right) + p$$

where $p \in \mathbb{R}^2$ and

$$\theta(s) = \int \kappa(s) ds + \theta_0$$

with $\theta_0 \in \mathbb{R}$.

  - *Hint*: Use the fact that $T = \gamma'$ has unit length hence has the form $T = (\cos\theta(s), \sin\theta(s))$ for some smooth function θ (the implicit function theorem guarantees smoothness). Now determine N in terms of T and differentiate T to obtain an equation for θ in terms of κ. Then finally, integrate T to obtain γ.

- Exercise: Conclude that all arc-length parametrisations of the unit circle centered on the origin are of the form

$$\gamma(s) = (\cos(s + s_0), \sin(s + s_0)).$$

Invariance Under Rigid Motion

Definition

A *rigid motion* of the plane is any affine transformation

$$T(x) = A \cdot x + b, \quad x \in \mathbb{R}^2$$

$b \in \mathbb{R}^2$ and where A is an *orthogonal matrix*. That is

$$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^2.$$

- Exercise: Let γ be a regular parametrised curve and define a new regular parametrised curve $\mu(s) = T(\gamma(s)) = A \cdot \gamma(s) + b$.
    1. Show that $T_\mu = A \cdot T_\gamma$ and $N_\mu = \pm A \cdot N_\gamma$ (the sign depends on whether A preserves or reverses orientation).
    2. Show that if γ is parametrised by arc-length, then so is μ.
    3. Show that $\kappa_\mu(s) = \kappa_\gamma(s)$.

We say that κ is *invariant under rigid motion*.

Lecture Three: Curves - Space Curves

Normal and Binormal Vectors

A regular, parametrised *space curve* is a smooth map $\gamma : (a, b) \to \mathbb{R}^3$ with $\gamma' \neq 0$.
Unlike for plane curves, we cannot a-priori define a normal vector: Putting $T = \gamma' / |\gamma'|$,

$$T^{\perp}(t) = \{V \in \mathbb{R}^2 : \langle T(t), V \rangle = 0\}$$

is a two-dimensional plane passing through $\gamma(t)$.
As with plane curves however we may still parametrise by arc-length and then

$$1 \equiv \langle T, T \rangle \Rightarrow \gamma'' \perp T.$$

Therefore, if $\gamma'' \neq 0$, we may choose a unit normal vector N in $T^{\perp}$ and a *binormal vector* $B \in T^{\perp}$ to obtain an oriented basis $\{T, N, B\}$ of $\mathbb{R}^3$:

$$T(s) = \gamma', \quad N(s) = \frac{\gamma''}{|\gamma''|}, \quad B(s) = T(s) \times N(s).$$

Curvature and Torsion

We define the curvature,

$$\kappa(s) = \left|\gamma''(s)\right|.$$

For space curves, we will restrict to the curves with $\kappa > 0$ so that $N = \frac{\gamma'}{\kappa}$ is defined.

Here we cannot give a sign to the curvature since we cannot a-priori choose a normal vectors.

Since B is unit length, $\partial_s B \perp B$. Moreover

$$\partial_s B = \partial_s (T \times N) = T' \times N + T \times N' = T \times N' \perp T$$

since $T' = \kappa N \Rightarrow T' \times N = 0$.

Therefore we define the *torsion*, τ by

$$B' = -\tau N.$$

- Exercise: Since N is unit length, $\partial_s N \perp N$ and

$$\partial_s N = -\kappa T + \tau B$$

Frenet-Serret Frame

Now we have a three dimensional frame $\{T, N, B\}$.
The *Osculating Plane* is the plane spanned by T and N.
The Frenet-Serret equations are

$$\partial_s \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

- κ measures the deviation of γ from the tangent line in the osculating plane.
- τ measures the *twisting* of γ away from the osculating plane.
- A space curve γ with $\kappa > 0$ lies in a plane if and only if $\tau \equiv 0$.
- Given $\kappa > 0$ and τ, there exists a unique (up to rigid motion) curve with the given curvature and torsion.

Lecture Three: Curves - Global Results

Jordan Curve Theorem

Definition

Let $\gamma : (a, b) \to \mathbb{R}^2$ be a smooth curve. We say γ is *simple* if γ is one-to-one. We say γ is *closed* if $\gamma(a) = \gamma(b)$ and likewise $\gamma^{(k)}(a) = \gamma^{(k)}(b)$ for all $k \in \mathbb{N}$.

Theorem (Jordan Curve Theorem)

Let γ be a simple, closed curve. Then γ divides the plane into two regions - one bounded and one unbounded. That is, there exists two disjoint, connected open sets Ω and Λ with $\partial\Omega = \partial\Lambda = \gamma$ such that

1. *There exists an $R > 0$ such that $\Omega \subseteq B_R(0)$, and*
2. $\mathbb{R}^2 = \Omega \sqcup C \sqcup \Lambda$ *where* $C = \mathrm{Img}(\gamma)$ *and the union is a disjoint union.*

Remark

Necessarily, Λ is *unbounded* in the sense that Λ is not contained in any $B_R(0)$.

Proofs of the Jordan Curve Theorem

- The hard part of the theorem is that it applies to *continuous curves*.
- Thus γ could be for example *nowhere differentiable* such as a *fractal*.
- A proof in the piecewise smooth case (i.e. where γ is continuous and smooth away from at most finitely many points) can be found here: The Jordan Curve Theorem for Piecewise Smooth Curves, R. N. Pederson, The American Mathematical Monthly Vol. 76, No. 6 (Jun. - Jul., 1969), pp. 605-610

  The idea is that γ is *locally two-sided*: the tangent line divides the plane into the two sides. Thus one normal points inward while the other points outward.
- A reasonably elementary general proof may be found here: The Jordan Curve Theorem Via the Brouwer Fixed Point Theorem, Ryuji Maehara, The American Mathematical Monthly, Vol. 91, No. 10 (Dec., 1984), pp. 641-643
- More generally the theorem holds for *embedded spheres* $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$.
- See `https://en.wikipedia.org/wiki/Jordan_curve_theorem` for more details.

Isoperimetric Inequality

- Let γ be a simple, closed curve and Ω the bounded region enclosed by γ.

Theorem (Isoperimetric Inequality)

We have

$$\frac{L^2}{A} \geq 4\pi$$

where L is the length of γ and A is the enclosed area, i.e. the area of Ω. Moreover, equality occurs if and only if γ is a (round) circle.

- The circle encloses the most area for a given perimeter. Equivalently, the circle has the least perimeter for a given area.
- Look up "Queen Dido"!

Isoperimetric Inequality

Proof.

Recall the *Divergence Theorem*:

$$\int_\Omega \operatorname{div} X \, dxdy = \int_\gamma \langle X, N \rangle ds.$$

Let $X(x,y) = (x,y)$ so that $\operatorname{div} X = \partial_x x + \partial_y y = 2$.
Then

$$\begin{aligned} 2A = \int_\gamma \langle X, N \rangle ds &\leq \int_\gamma |X| \, ds \quad \text{(Pointwise Cauchy Schwartz)} \\ &\leq \left(\int |X|^2 \, ds \right)^{1/2} \left(\int 1^2 ds \right)^{1/2} \quad (L^2 \text{ Cauchy Schwartz}) \\ &= \left(\int |X|^2 \, ds \right)^{1/2} L^{1/2}. \end{aligned}$$

Isoperimetric Inequality

Proof.

Write $\gamma(s) = (x(s), y(s))$ and translate:

$$(x(s), y(s)) \mapsto (x(s) + u, y(s) + v).$$

Notice that:

1. L and A are invariant under translation.
2. $\lim_{u \to \pm\infty} x(s) + u = \pm\infty$ uniformly in s. Therefore there exists a u such that $\int x(s)ds = 0$. Likewise, there is a v such that $\int y(s)ds = 0$.

Then since x is periodic and $\int xds = 0$, we may apply *Wirtinger's Inequality*:

$$\int_0^L (x')^2 ds \geq \frac{4\pi^2}{L^2} \int_0^L x^2 ds$$

and likewise for y.

Isoperimetric Inequality

Proof.

Thus

$$\begin{aligned} 2A &\leq L^{1/2}\left(\int |X|^2\, ds\right)^{1/2} = L^{1/2}\left(\int x^2 + y^2 ds\right)^{1/2} \\ &\leq L^{1/2}\left(\frac{L^2}{4\pi^2}\int (x')^2 + (y')^2 ds\right)^{1/2} \quad \text{(Wirtinger)} \\ &= L^{1/2}\frac{L}{2\pi}L^{1/2} \quad \text{arc length: } (x')^2 + (y')^2 = 1 \\ &= \frac{L^2}{2\pi}. \end{aligned}$$

□

Theorem of Turning Tangents

Theorem (Turning Tangents)

Let γ be a simple, closed curve. Then

$$\int_\gamma \kappa ds = \pm 2\pi.$$

Proof.

Since $|T| \equiv 1$ we may write

$$T(s) = (\cos\theta(s), \sin\theta(s)).$$

By the implicit function theorem, the function θ is smooth.
By the chain rule and the Frenet-Serret formula

$$\theta'(-\sin\theta, \cos\theta) = \partial_s T = \kappa N.$$

Theorem of Turning Tangents

Proof.

But $N = (-\sin\theta, \cos\theta)$ and hence $\theta' = \kappa$.
Then

$$\int_0^L \kappa ds = \int_0^L \theta'(s) ds = \theta(L) - \theta(0).$$

Since γ is closed, $(\cos\theta(L), \sin\theta(L)) = (\cos\theta(0), \sin\theta(0))$. Therefore

$$\int_0^L \kappa ds = \theta(L) - \theta(0) = 2\pi n$$

for some integer $n \in \mathbb{Z}$.
The integer n is known as the *winding number* of γ. A topological result says for a simple closed curve $n = \pm 1$ with the sign depending on the orientation. □