MATH704 Differential Geometry

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Lecture Four: Multivariable Calculus Refresher

1. Lecture Four: Multivariable Calculus Refresher
   - Topology on $\mathbb{R}^n$
   - Limits and continuity
   - Differentiability
   - Inverse and Implicit Function Theorems

Lecture Four: Multivariable Calculus Refresher - Topology on $\mathbb{R}^n$

Open and closed balls and spheres

Definition

Given $r > 0$ and $x \in \mathbb{R}^n$, the *open ball* of radius r and centre x is the set

$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

The *closed ball* of radius r and centre x is the set

$$\bar{B}_r(x) = \{y \in \mathbb{R}^n : |x - y| \leq r\}.$$

The *sphere* of radius r and centre x is the set

$$\mathbb{S}_r(x) = \{y \in \mathbb{R}^n : |x - y| = r\}.$$

Distance function

Recall the distance $|x - y|$ is defined to be

$$|x - y| = \sqrt{(x^1 - y^1)^2 + \cdots + (x^n - y^n)^2}$$

where $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$.

- The open ball is the set of points of distance to x strictly less than r.
- The closed ball is the set of points of distance to x less than or equal to r.
- The sphere is the set of points of distance to x equal to r.

It is sometimes said that analysis is simply applications of the triangle inequality:

$$|x - y| \leq |x - z| + |z - y| .$$

Open and closed sets

Definition

A set $U \subset \mathbb{R}^n$ is said to be *open* provided for every $x \in U$, there exists an $r = r(x)$ such that

$$B_r(x) \subseteq U.$$

A set C is *closed* if it's complement,

$$\mathbb{R}^n \setminus C := \{y \in \mathbb{R}^n : y \notin C\}$$

is open.

- By this definition, open balls are open, closed balls are closed and spheres are closed.
- Given any point of an open set, we can always move /uniformly/ a little in any direction and remain in the open set.

Bounded and compact sets

Definition

A set $S \subseteq \mathbb{R}^n$ is *bounded* if there exists an $x \in \mathbb{R}^n$ and an $r > 0$ such that $S \subseteq B_r(x)$.
A set $K \subseteq \mathbb{R}^n$ is *compact* if it is closed and bounded.

- A set $S \subseteq \mathbb{R}^n$ is bounded if and only if, for every $x \in \mathbb{R}^n$ there exists an $r = r(x)$ such that $S \subseteq B_r(x)$. This follows by the triangle inequality.
- A set $K \subseteq \mathbb{R}^n$ is compact if and only if for every *open cover* $\{U_\alpha\}$, there exists a *finite subcover*.
  - An *open cover* is a collection of open sets $\{U_\alpha\}$ such that $K \subseteq \cup_\alpha U_\alpha$.
  - A *finite subcover* is a finite number of sets $U_{\alpha_1}, \cdots, U_{\alpha_N}$ from the collection such that $K \subseteq \cup_{i=1}^N U_{\alpha_i}$.
  - This equivalent condition of compactness is the *general definition for topological spaces* but is equivalent in the case of $\mathbb{R}^n$.

Lecture Four: Multivariable Calculus Refresher - Limits and continuity

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   - Differentiability
   - Inverse and Implicit Function Theorems

Limits

Definition

A sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^n$ *converges* to $x \in \mathbb{R}^n$ if for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $(x_n)_{n\geq N} \subseteq B_\epsilon(x)$. We write $\lim_{n\to\infty} x_n = x$.

Definition

The sequence (x_n) is *Cauchy* if for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $(x_m)_{m\geq N} \subseteq B_\epsilon(x_n)$ for every $n \geq N$.

Remark

The condition for convergence to x says that $|x - x_n| < \epsilon$ for $n \geq N$. The condition to be a Cauchy sequence says that $|x_n - x_m| < \epsilon$ for $m, n \geq N$.

Continuity

Here are some equivalent definitions of continuity.

Definition (Sequential definition)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x \in \mathbb{R}^n$ if for *every* sequence (x_n) with $\lim_{n\to\infty} = x$ we have $\lim_{n\to\infty} f(x_n) = f(x)$.

Definition (ϵ-δ definition)

Write

$$\lim_{x\to x_0} f(x) = y$$

provided for every $\epsilon > 0$, there exists a $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(y)$. Then f is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$.

Definition (Topological definition)

The function f is continuous (at every x_0) if $f^{-1}(V)$ is an open set for every open set $V \subseteq \mathbb{R}^m$.

Continuity

- The first definition requires that $f(x_n) \to f(x)$ for *every* sequence.
- The condition in the second definition that $f(B_\delta(x_0)) \subseteq B_\epsilon(y)$ is the same thing as $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.
- The second definition says that given *any tolerance* $\epsilon > 0$, there is an *adjustment* $\delta > 0$ so that provided we are sufficiently close to x_0 (i.e. $|x - x_0| < \delta$), then $f(x)$ is within the desired tolerance of $f(x_0)$ (i.e. $|f(x) - f(x_0)| < \epsilon$.
- The equivalence of the first and second definitions is a standard exercise in analysis using the *completeness* of the real numbers $\mathbb{R}$.
- The final definition is the general *topological* definition.
- The equivalence of the topological and ϵ-δ definitions follows by writing $U = \cup_{y \in U} B_{r(y)}(y)$ as a union of open balls and using properties of the pull-back f^{-1}.

A cautionary example

Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Then f is not continuous at $(x,y) = (0,0)$.
However, along every straight line through the origin $y = ax$, the limit is in fact 0! That is,

$$\lim_{t\to 0} f(t, at) = \lim_{t\to 0} \frac{t^2 \cdot at}{t^4 + a^2t^2} = \lim_{t\to 0} \frac{t^2}{t^2}\frac{at}{t^2 + a^2} = 0.$$

But along the curve $y = x^2$, we get something else:

$$\lim_{t\to 0} f(t, t^2) = \lim_{t\to 0} \frac{t^2 \cdot t^2}{t^4 + (t^2)^2} = \lim_{t\to 0} \frac{t^4}{t^4}\frac{1}{2} = \frac{1}{2}.$$

Lecture Four: Multivariable Calculus Refresher - Differentiability

Partial derivatives

Definition

The i'th *partial derivative* of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $x = (x^1, \ldots, x^n)$ is

$$\partial_i f(x) = \frac{\partial f}{\partial x^i}(x) = \lim_{h \to 0} \frac{f(x^1, \ldots, x^{i-1}, x^i + h, x^{i+1}, \ldots x^n) - f(x^1, \ldots, x^n)}{h}.$$

whenever the limit exists.

The partial derivative is simply the usual derivative of a function of one variable holding all other variables fixed.

Directional derivatives

Definition

Let $X = (X^1, \ldots, X^n) \in \mathbb{R}^n$. The *directional derivative* $df_x \cdot X$ of f at x in the direction X is

$$\partial_X f(x) = \partial_t|_{t=0} f(x + tX) = \lim_{h \to 0} \frac{f(x + hX) - f(x)}{h}.$$

The partial derivative is simply the directional derivative with $X = e_i$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the i'th position is the so-called i'th basis vector.

The Differential

Recall that Taylor's theorem with remainder states that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R_{x_0}(x)$$

where

$$\lim_{x \to x_0} \frac{|R_{x_0}(x)|}{x - x_0} = 0.$$

We write $R_{x_0}(x) = o(x)$ as $x \to x_0$.

Definition

We say $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x_0 if there exists a linear map $L_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - L_{x_0} \cdot (x - x_0)|}{|x - x_0|} = 0.$$

That is, there exists a linear map written $L_{x_0} = df_{x_0}$ such that

$$f(x) = f(x_0) + df_{x_0} \cdot (x - x_0) + o(|x - x_0|), \quad \text{as } x \to x_0.$$

Differentiable imples partial derivatives exist

Let f be differentiable at $x_0 = (x_0^1, \ldots, x_0^n)$. For $h \neq 0$, let $x = (x_0^1, \ldots, x_0^{i-1}, x_0^i + h, x_0^{i+1}, \ldots, x_0^n) = x_0 + he_i$. We have

$$\partial_i f(x_0) = \lim_{h \to 0} \frac{f(x_0 + he_i) - f(x_0)}{h}$$

provided the limit exists. Differentiability ensures that

$$0 = \lim_{h \to 0} \left| \frac{f(x_0 + he_i) - f(x_0)}{h} - \frac{df_{x_0} \cdot he_i}{h} \right|$$

and hence

$$\lim_{h \to 0} \frac{f(x_0 + he_i) - f(x_0)}{h} = \lim_{h \to 0} \frac{1}{h} df_{x_0} \cdot he_i = df_{x_0} \cdot e_i.$$

exists.

- Exercise: Show that the same argument proves $\partial_X f(x_0) = df_{x_0}(X)$ exists.

A cautionary example

Let

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Notice that

$$\partial_x f(0,0) = \partial_t|_{t=0} f(t,0) = \partial_t|_{t=0} \frac{t \cdot 0}{t^2 + 0^2} = 0.$$

Likewise $\partial_y f(0,0) = 0$.
However,

$$\partial_{(1,1)} f(0,0) = \partial_t|_{t=0} f(t,t) = \lim_{t \to 0} \frac{1}{t}(f(t,t) - f(0,0))$$

is undefined since $f(t,t) = t^2/(t^2+t^2) = 1/2$.
Defining, $f(0,0) = 1/2$ doesn't help because then $\partial_{(1,2)} f(0,0)$ doesn't exist. In fact, f is not even continuous at $(0,0)$.

C^1 functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1 (i.e. has continuous derivative) if f is differentiable at each x and moreover, the map

$$x \mapsto df_x$$

is continuous. This is equivalent to having /continuous/ partial derivatives.

Note here that df_x is a linear map $\mathbb{R}^n \to \mathbb{R}^m$ and the set of all these is linearly isomorphic to the space $M_{n,m}$ of n by m matrices, which is itself linearly isomorphic to $\mathbb{R}^{nm}$ (index by i, j with $1 \leq i \leq n$ and $1 \leq j \leq m$). Concretely we may realise df_x as the matrix

$$(df_x)_{ij} = \partial_i f^j(x) \quad \text{since} \quad df_x \cdot e_i = \partial_i f(x) = (\partial_i f^1, \ldots, \partial_i f^n).$$

Then $df : \mathbb{R}^n \to \mathbb{R}^{nm}$ is a map between Euclidean spaces so we can ask if it's differentiable. Then f is C^2 if $d^2 f$ is C^1 and more generally, f is C^k if $d^k f$ is continuous.

Chain Rule

Theorem (Chain Rule)

The chain rule states that if $f : \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable at* x_0 *and* $g : \mathbb{R}^m \to \mathbb{R}^k$ *is differentiable at* $f(x_0)$*, then*

$$d(f \circ h)_{x_0} = dh_{f(x_0)} \cdot df_{x_0}.$$

By the *chain rule*, given any curve γ such that $\gamma(0) = x$ and $\gamma'(0) = X$ we have

$$df_x \cdot X = \partial_t|_{t=0} f(\gamma(t)).$$

In other words, to compute $\partial_X f(x)$ we may replace the curve $t \mapsto x + tX$ with any other curve such that $\gamma(0) = x$ and $\gamma'(0) = X$.

Lecture Four: Multivariable Calculus Refresher - Inverse and Implicit Function Theorems

One Dimensional Inverse Function Theorem

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with $f'(x_0) \neq 0$, there exists an interval I containing x_0 and an interval J containing $f(x_0)$ so that $f : I \to J$ is a diffeomorphism. That is, there exists an inverse function $f^1 : J \to I$. Moreover, for all $y \in J$,

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

- To be explicit, the definition of f^{-1} means that $f \circ f^{-1}(y) = y)$ for all $y \in J$ and $f^{-1} \circ f(x) = x$ for all $x \in I$.
- In this case, observe that if $h : J \to \mathbb{R}$ is a smooth function, then so too is $h \circ f$. This defines the *pull-back*

$$f^* : h \in C^\infty(J, \mathbb{R}) \mapsto h \circ f \in C^\infty(I, \mathbb{R}).$$

- Exercise: Show that f^* is a bijection with inverse $(f^{-1})^*$.

Contraction mappings and fixed points

Definition

A map $T : \bar{B}_r(p) \to \bar{B}_r(p)$ is a *contraction map* if there exists a constant $0 \leq L < 1$ such that

$$|T(x) - T(y)| \leq L\,|x - y|\,.$$

Theorem (Banach fixed point theorem)

Let T be a contraction map. Then there exists a unique fixed point $x^* \in B_r(p)$ of T. That is, there exists a unique point x^* such that $T(x^*) = x^*$.

Proof of contraction mapping theorem (Uniqueness)

Proof.

Fundamental contraction identity:

$$
\begin{aligned}
|x - y| &\leq |x - T(x)| + |T(x) - y| \\
&\leq |x - T(x)| + |T(x) - T(y)| + |T(y) - y| \\
&\leq |x - T(x)| + L\,|x - y| + |T(y) - y|\,.
\end{aligned}
$$

Therefore

$$
|x - y| \leq \frac{|x - T(x)| + |T(y) - y|}{1 - L}
$$

Thus we obtain *uniqueness*: if $T(x) = x$ and $T(y) = y$, then $|x - y| \leq 0$ and hence $x = y$.

Proof of contraction mapping theorem (Existence)

Proof.

Pick any x_0 and define $x_n = T^n(x_0) = \underbrace{T \circ \cdots \circ T}_{n \text{ times}}(x_0)$

The claim is that $x^* = \lim_{n\to\infty} x_n$ exists and is the desired fixed point. Supposing first that the limit exists, then using $x_n = T(x_{n-1})$ we have

$$x_* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} x_{n-1}) = T(x^*)$$

where we pass the limit through T since a contraction mapping is continuous (for any ϵ choose $\delta = \epsilon/L$).

Proof of contraction mapping theorem (Existence)

Proof.

To prove that $x_n = T^n(x_0)$ has a limit we prove it's a Cauchy sequence. By the fundamental contraction identity

$$\begin{aligned}
|T^n(x_0) - T^m(x_0)| &\leq \frac{|T(T^n(x_0)) - T^n(x_0)| + |T(T^m(x_0)) - T^m(x_0)|}{1-L} \\
&= \frac{|T^n(T(x_0)) - T^n(x_0)| + |T^m(T(x_0) - T^m(x_0)|}{1-L} \\
&\leq \frac{L^n |T(x_0) - x_0| + L^m |T(x_0) - x_0|}{1-L} \\
&= \frac{L^n + L^m}{1-L} |T(x_0) - x_0| \to 0 \quad \text{as } n, m \to \infty.
\end{aligned}$$

Here we used that $0 \leq L < 1$ and by induction that (exercise!)

$$|T^n(x) - T^n(y)| \leq L^n |x - y|.$$

Inverse Function Theorem

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a smooth function such that df_{x_0} is invertible at x_0. Then there is an open set U containing x_0 and an open set V containing $f(x_0)$ such that $f|U : U \to V$ is a diffeomorphism. Moreover

$$df^{-1}_{f(x_0)} = (df_{x_0})^{-1}.$$

Remark

Notice that if f is a diffeomorphism, then $f^{-1} \circ f(x) = x$. That is, $f^{-1} \circ f = \mathrm{Id}_x$. Since $d\,\mathrm{Id}_x = \mathrm{Id}_n$, by the chain rule we have

$$\mathrm{Id}_n = d\,\mathrm{Id}_x = d(f^{-1} \circ f)_{x_0} = df^{-1}_{f(x_0)} \cdot df_{x_0}.$$

That is df_{x_0} is invertible and

$$(df_{x_0})^{-1} = df^{-1}_{f(x_0)}.$$

Inverse Function Theorem: Idea

Proof.

Here's the basic idea: By definition, we have

$$f(x) = f(x_0) + df_{x_0} \cdot (x - x_0) + o(|x - x_0|).$$

Ignoring the error term for the moment, by assumption since df_{x_0} is invertible, we can solve *uniquely* for x:

$$f(x) = f(x_0) + df_{x_0} \cdot (x - x_0) \quad \Rightarrow \quad x = x_0 + df_{x_0}^{-1}(f(x) - f(x_0)).$$

Write $y = f(x)$ and $y_0 = f(x_0)$. Since y uniquely determines x we may write $x = f^{-1}(y)$ and

$$f^{-1}(y) = f^{-1}(y_0) + df_{x_0}^{-1} \cdot (y - y_0).$$

So we need to deal with the error terms.

Inverse Function Theorem: Contraction

Proof.

We use the contraction mapping theorem: Define for each fixed y,

$$T_y(x) = x - df_{x_0}^{-1}(f(x) - y).$$

Then since f is C^1, so too is T (dropping the y subscript for convenience) and

$$dT_{x_0} = d\,\mathrm{Id}_{x_0} - df_{x_0}^{-1} df_{x_0} = 0.$$

By continuity of dT, there exists an open neighbourhood U of x_0 such that $\|dT_{x_0}\| \leq 1/2$. That is, for $x \in U$ and $X \in \mathbb{R}^n$,

$$|dT_x \cdot X| \leq \frac{1}{2}|X|.$$

Inverse Function Theorem: Contraction

Proof.

From $|dT_x \cdot X| \leq \frac{1}{2}|X|$, and the mean value inequality, we obtain

$$|T(x_1) - T(x_2)| \leq \frac{1}{2}|x_1 - x_2|$$

so that T is *contractive* for $x_1, x_2 \in U$.
In order to conclude that T has a unique fixed point, we need to verify that there is an $r > 0$ such that $T : \bar{B}_r(x_0) \to \bar{B}_r(x_0)$.
Since $x_0 \in U$ and U is open, there exists an $r > 0$ such that $B_r(x_0) \subseteq U$.

Inverse function theorem: Contraction

Proof.

Now we restrict the range of possible y: Let $y_0 = f(x_0)$ and $y \in B_s(y_0)$ with s any number satisfying

$$0 < s < \frac{1-L}{\|df_{x_0}^{-1}\|} r.$$

Then for $x \in B_r(x_0)$, recalling $T(x) = x - df_{x_0}^{-1}(f(x) - y)$ we have

$$\begin{aligned} |T(x) - x_0| &\leq |T(x) - T(x_0)| + |T(x_0) - x_0| \\ &\leq L|x - x_0| + \left|-df_{x_0}^{-1}(f(x_0) - y)\right| \\ &\leq L|x - x_0| + \|df_{x_0}^{-1}\| \, |y_0 - y| \\ &\leq rL + \|df_{x_0}^{-1}\| s \\ &\leq rL + (1 - L)r = r. \end{aligned}$$

Inverse function theorem: Fixed Point

Proof.

That is $T(x) \in \bar{B}_{x_0}(r)$ for $x \in \bar{B}_{x_0}(r)$ and $y \in \bar{B}_s(y_0)$.
Thus for any $y \in \bar{B}_s(x_0)$, $T_y : \bar{B}_r(x_0) \to \bar{B}_r(x_0)$ is a contraction mapping, hence:
For each such y, there exists a unique fixed point $x^*_y \in \bar{B}_r(x_0)$. That is

$$x^*_y = T_y(x^*_y) = x^*_y - df^{-1}_{x_0}(f(x^*_y) - y).$$

Cancelling x^*_y from both sides and since $df^{-1}_{x_0}$ is non-singular,

$$df^{-1}_{x_0}(f(x^*_y) - y) = 0 \Rightarrow f(x^*_y) = y.$$

Inverse function theorem: Continuity of Inverse

Proof.

We have finally found our inverse function: $f^{-1}(y) = x_y^*$ for $y \in B_s(y_0)$.
Note we need to restrict the range of x to the open set $f^{-1}(B_s(y_0)) \cap B_r(x_0)$ so that f maps this set into $B_s(y_0)$.
Since T is a contraction

$$\left|x_1 - x_2 - df_{x_0}^{-1}(f(x_1) - f(x_2))\right| = |T(x_1) - T(x_2)| \leq L\,|x_1 - x_2|\,.$$

By the *reverse triangle inequality*

$$|x_1 - x_2| - \left|df_{x_0}^{-1}(f(x_1) - f(x_2))\right| \leq L\,|x_1 - x_2|\,.$$

That is,

$$|x_1 - x_2| \leq \frac{\left|df_{x_0}^{-1}(f(x_1) - f(x_2))\right|}{1 - L} \leq \frac{\|df_{x_0}^{-1}\|}{1 - L}\,|f(x_1) - f(x_2)|\,.$$

Inverse function theorem: Continuity of Inverse

Proof.

We have

$$|x_1 - x_2| \leq \frac{\|df_{x_0}^{-1}\|}{1 - L} |f(x_1) - f(x_2)| .$$

Letting $y_i = f(x_i)$ so that $x_i = f^{-1}(y_i)$ gives continuity (even Lipschitz):

$$\left|f^{-1}(y_1) - f^{-1}(y_2)\right| \leq \frac{\|df_{x_0}^{-1}\|}{1 - L} |y_1 - y_2| .$$

Lipschitz is almost differentiable but not quite (e.g. $f(x) = |x|$).

Inverse function theorem: Differentiability

Proof.

Pick any arbitrary $y \in B_s(y_0)$ and any h such that $y + h \in B_s(y_0)$, say $h \in B_\epsilon(0)$ so that $y + h \in B_\epsilon(y) \subseteq B_s(y_0)$.
Let $x = f^{-1}(y)$ and define

$$R = f^{-1}(y + h) - f^{-1}(y) - df_x^{-1} \cdot h.$$

We need to show that

$$\lim_{h \to 0} \frac{|R|}{|h|} = 0.$$

Inverse function theorem: Differentiability

Proof.

Let $k = f^{-1}(y+h) - f^{-1}(y)$ so that $h = f(x+k) - f(x)$. Then

$$\begin{aligned} R &= f^{-1}(y+h) - f^{-1}(y) - df_x^{-1} \cdot h \\ &= k - df_x^{-1}(f(x+k) - f(x)) \\ &= k - df_x^{-1}(df_x k + o(k)) \\ &= -df_x^{-1}(o(k)). \end{aligned}$$

Inverse function theorem: Differentiability

Proof.

Since f^{-1} is Lipschitz, with constant M say, we have

$$|k| = \left|f^{-1}(y+h) - f^{-1}(y)\right| \leq M\,|y+h-y| = M\,|h|\,.$$

Therefore,

$$\frac{|R|}{|h|} \leq \|df_x^{-1}\|\frac{o(k)}{|h|} \leq M\|df_x^{-1}\|\frac{o(k)}{|k|}.$$

The right hand side goes to zero as $h \to 0$ since $|k| \leq M\,|h|$ implies $k \to 0$ and then by definition of $o(k)$.

Inverse function theorem: Higher regularity

Proof.

So to summarise we have shown the existence of a differentiable local inverse f^{-1} to f with differential

$$d(f^{-1})_y = (df_x)^{-1}$$

where $x = f^{-1}(y)$.
Now, by Cramers's rule, given an invertible matrix A, the inverse is

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

where the $\operatorname{adj} A$ is the *adjugate matrix* formed from cofactors of A - that is the determinants of the minors of A.
As a function then, $A \mapsto A^{-1}$ we see that the components are rational functions of the entries of A (since determinants are polynomials in the entries of A).

Inverse function theorem: Higher regularity

Proof.

Then the inverse function Inv is in fact a smooth function from the open set of non-singular matrices (i.e. those with $\det A \neq 0$) to itself.
Then since $x \mapsto df_x$ is smooth,

$$y \mapsto df^{-1}_{f^{-1}(y)} = \operatorname{Inv} \circ df \circ f^{-1}(y)$$

is the composition of C^0 functions and hence df^{-1} is also C^0.
That is f^{-1} is C^1. Therefore in fact df^{-1} is the composition of C^1 functions hence is also C^1.
That is f^{-1} is C^2. Now we just iterate to get f^{-1} is C^k for any k and hence smooth. □

Implicit Function Theorem

- *In progress.*

Submersions and Immersions

- *In progress.*