MATH704 Differential Geometry

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Lecture Seven: Geometry Of Regular Surfaces EXPORT

Lecture Seven: Geometry Of Regular Surfaces EXPORT - Smooth maps, differentials and tangent vectors

Smooth Curves

Definition

A curve, $\gamma : (a, b) \to S$ is *smooth* if for every local parametrisation, $\varphi : U \to S$, the curve

$$\varphi^{-1} \circ \gamma : \gamma^{-1}(\varphi(U)) \to U$$

is smooth for all $t \in (a, b)$ such that $\gamma(t) \in \varphi(U)$.

It is sufficient that $\varphi_\alpha^{-1} \circ \gamma$ is smooth for any cover $\{\varphi_\alpha : U_\alpha \to V_\alpha \subseteq S\}_{\alpha \in \mathcal{A}}$ of the image $\gamma((a, b))$.

- If $\varphi : U \to S$ is any parametrisation such that $\gamma(t) \in \varphi(U)$, then choose any α such that $\gamma(t) \in V_\alpha$.
- The transition map $\tau = \varphi^{-1} \circ \varphi_\alpha$ is smooth. Therefore

$$\varphi^{-1} \circ \gamma(t) = \varphi^{-1} \circ \varphi_\alpha \circ \varphi_\alpha^{-1} \circ \gamma = \tau \circ (\varphi_\alpha^{-1} \circ \gamma)$$

  is smooth.

Coordinate Curves

Every curve $\mu : (a, b) \to U$ gives a smooth curve $\gamma = \varphi \circ \mu : (a, b) \to S$.
Just observe that

$$\varphi^{-1} \circ \gamma = \varphi^{-1} \circ \varphi \circ \mu = \mu$$

is smooth.
For example, we have the smooth *coordinate curves* through (u_0, v_0)):

$$\gamma_u(t) := \varphi(t, v_0)$$

where $t \in (u_0 - \epsilon, u_0 + \epsilon)$ for some $\epsilon > 0$ so that $(t, v_0) \in U$. Similarly,

$$\gamma_v(t) := \varphi(u_0, t)$$

Smooth Curves

Lemma

A curve $\gamma : (a, b) \to S$ is smooth if and only if it is smooth as a map $\gamma : (a, b) \to \mathbb{R}^3$.

Proof.

The observation is that by the inverse function theorem, any local parametrisation $\varphi : U \to S$ extends to a smooth diffeomorphism

$$\Phi : W \subseteq_{\text{open}} U \times \mathbb{R} \to Z \subseteq_{\text{open}} \mathbb{R}^3$$

with $\Phi(u, v, 0) = \varphi(u, v)$.
Then $\varphi^{-1} \circ \gamma = \Phi^{-1} \circ \gamma$.
Exercise: Fill in the details! □

Smooth Maps

Definition

Let S_1, S_2 be regular surfaces. A map $f : S_1 \to S_2$ is *smooth* if

$$\psi \circ f \circ \varphi^{-1} : \varphi[f^{-1}[Z] \cap V] \subseteq U \subseteq \mathbb{R}^2 \to W \subseteq \mathbb{R}^2$$

is smooth for every pair of local parametrisations

$$\varphi : U \to V \subseteq S, \quad \psi : W \to Z \subset S'$$

Again, if f is smooth with respect to one pair of parametrisations, then it is smooth with respect to all overlapping ones:

$$\begin{aligned}\psi_2 \circ f \circ \varphi_2^{-1} &= \psi_2 \circ (\psi_1^{-1} \circ \psi_1) \circ f \circ (\varphi_1^{-1} \circ \varphi_1) \circ \varphi_2^{-1} \\ &= (\psi_2 \circ \psi_1^{-1}) \circ (\psi_1 \circ f \circ \varphi_1^{-1}) \circ (\varphi_1 \circ \varphi_2^{-1}) \\ &= \tau_{21}^{\psi} \circ \psi_1 \circ f \circ \varphi_1^{-1} \circ \tau_{12}^{\varphi}.\end{aligned}$$

is smooth provided $\psi_1 \circ f \circ \varphi_1^{-1}$ is smooth.

Tangent Plane

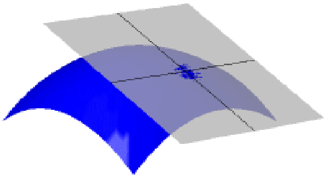
Definition

Let $x \in S$. The tangent plane T_xS to S at x consists of all the vectors $X \in \mathbb{R}^3$, based at x and tangent to S.

Equivalent Descriptions

- Velocity vectors: $T_xS = \{\gamma'(0) | \gamma : I \to S, \gamma(0) = x\}$
- Image of the differential: $T_xS = \{d\varphi_0(X) | \varphi : U \to S, \varphi(0) = x\}$

The second definition is independent of the choice of parametrisation!



The Differential

Definition

Let $f : S \to S'$ be a smooth map. The differential, df_x of f at $x \in S$ is the linear map

$$\begin{aligned} df_x : T_x S &\to T_{f(x)} S' \\ \gamma'(0) &\mapsto (f \circ \gamma)'(0). \end{aligned}$$

Coordinate Description

Let $F(u, v) = \psi^{-1} \circ f \circ \varphi(u, v) = (F_1(u, v), F_2(u, v))$ with $x = f(u_0, v_0)$:

$$df_x = \begin{pmatrix} \frac{\partial F_1}{\partial u}(v_0, u_0) & \frac{\partial F_1}{\partial v}(v_0, u_0) \\ \frac{\partial F_2}{\partial u}(v_0, u_0) & \frac{\partial F_2}{\partial v}(v_0, u_0) \end{pmatrix}$$

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Metric (First Fundamental Form)

Definition

The metric g on S is an inner product g_x at each point $x \in S$ defined for tangent vectors $V = \gamma'(0), W = \mu'(0) \in T_xS \subseteq \mathbb{R}^3$ by

$$g_x(V, W) = \left\langle \gamma'(0), \mu'(0) \right\rangle_{\mathbb{R}^3}.$$

Equivalently

$$\begin{aligned} g(V, W) &= \left\langle c_1 \frac{\partial \varphi}{\partial x_1} + c_2 \frac{\partial \varphi}{\partial x_2}, d_1 \frac{\partial \varphi}{\partial x_1} + d_2 \frac{\partial \varphi}{\partial x_2} \right\rangle \\ &= c_1 d_1 \left\langle \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_1} \right\rangle + c_2 d_2 \left\langle \frac{\partial \varphi}{\partial x_2}, \frac{\partial \varphi}{\partial x_2} \right\rangle \\ &\quad + (c_1 d_2 + c_2 d_1) \left\langle \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2} \right\rangle \\ &= c_1 d_1 g_{11} + c_2 d_2 g_{22} + (c_1 d_2 + c_2 d_1) g_{12}. \end{aligned}$$

Local Coordinate Expression

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} := \begin{pmatrix} \left\langle \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_1} \right\rangle & \left\langle \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2} \right\rangle \\ \left\langle \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_1} \right\rangle & \left\langle \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_2} \right\rangle \end{pmatrix}$$

- This expression is *only valid in a local coordinate parametrisation* φ.
- The local matrix (g_{ij}) is *positive definite*.

Change of Local Coordinates

- Changing coordinates by $\varphi \mapsto \varphi \circ \tau$ where $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ leads to the change of coordinates for the metric

$$g_{ab}^{\varphi \circ \tau} = \langle \partial_{y^a}(\varphi \circ \tau), \partial_{y^b}(\varphi \circ \tau) \rangle = \left\langle \sum_i \partial_{x^i}\varphi \partial_{y^a}\tau^i, \sum_j \partial_{x^j}\varphi \partial_{y^b}\tau^j \right\rangle$$
$$= \sum_{ij} g_{ij} \partial_{y^a}\tau^i \partial_{y^b}\tau^j$$

where $\tau = (\tau^1(y^1, y^2), \tau^2(y^1, y^2))$ and $\varphi = \varphi(x^1, x^2)$.

- More concisely,

$$\begin{aligned} g^{\varphi \circ \tau}(X, Y) &= \langle d(\varphi \circ \tau) \cdot X, d(\varphi \circ \tau) \cdot Y \rangle \\ &= \langle d\varphi(d\tau \cdot X), d\varphi(d\tau \cdot y) \rangle \\ &= g^{\varphi}(d\tau \cdot X, d\tau \cdot Y). \end{aligned}$$

More on Coordinate Changes

Let $\varphi : U \to S$ and $\psi : V \to S$ be local parametrisations and $\tau = \varphi^{-1} \circ \psi$ the transition map.
Let X be a tangent vector to S so that $X = \gamma'(0)$ for some curve $\gamma : (-\epsilon, \epsilon) \to S \subseteq \mathbb{R}^3$.
Define the corresponding (smooth!) curves in the coordinate space:

$$\gamma_\varphi = \varphi^{-1} \circ \gamma, \quad \gamma_\psi = \psi^{-1} \circ \gamma.$$

Then we may write uniquely,

$$\gamma'_\varphi(0) = X^u_\varphi e_u + X^v_\varphi e_v, \quad \gamma'_\psi(0) = X^r_\psi e_r + X^s_\psi e_s.$$

More on Coordinate Changes

Notice that we have

$$\varphi \circ \gamma_\varphi = \varphi \circ (\varphi^{-1} \circ \gamma) = \gamma, \quad \psi \circ \gamma_\psi = \gamma.$$

Then recalling $X = \gamma'(0)$, $X^u_\varphi e_u + X^v_\varphi e_v = \gamma'_\varphi(0)$:

$$d\varphi \left(X^u_\varphi e_u + X^v_\varphi e_v\right) = \partial_t|_{t=0}\varphi(\gamma_\varphi(t)) = \gamma'(0) = X = d\psi \left(X^u_\psi e_u + X^v_\psi e_v\right).$$

Then since $\tau = \psi^{-1} \circ \varphi$, we have $\varphi = \psi \circ \tau$ and hence

$$d\psi \left(X^u_\psi e_u + X^v_\psi e_v\right) = X = d\varphi \left(X^u_\varphi e_u + X^v_\varphi e_v\right) = d\psi \cdot d\tau \left(X^u_\varphi e_u + X^v_\varphi e_v\right).$$

But $d\psi$ is injective hence we see how tangent vectors transform under change of coordinates (compare $g^{\varphi\circ\tau}(X, Y) = g^\varphi(d\tau \cdot X, d\tau \cdot Y)$):

$$X^u_\psi e_u + X^v_\psi e_v = d\tau \left(X^u_\varphi e_u + X^v_\varphi e_v\right).$$

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Length and Angle of Tangent Vectors

Definition

Let X be a tangent vector. Then it's length is defined to be

$$|X|_g := \sqrt{g(X, X)}.$$

Definition

The angle, θ between two tangent vectors X, Y (at the same point $x \in S$!) is defined by

$$\cos\theta = \frac{g(X, Y)}{|X|\,|Y|} = g\left(\frac{X}{|X|}, \frac{Y}{|Y|}\right).$$

.

Cauchy Schwartz Inequality

Lemma

$$|g(X, Y)| \leq |X|\,|Y|\,.$$

See https://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality#First_proof

Rearranging Cauchy-Schwarz inequality for $X, Y \neq 0$ gives

$$\frac{g(X, Y)}{|X|\,|Y|} \in [-1, 1]$$

and θ is well defined after choosing an inverse arccos.
The simplest is to take $\theta \in [0, \pi]$.

Arc Length

Definition

Let $\gamma : (a, b) \to S$ be a smooth curve. The *arc-length* of γ is

$$L(\gamma) = \int_a^b \left|\gamma'(t)\right| dt.$$

As for plane and space curves, define the arc length parameter

$$s(t) = \int_a^t \left|\gamma'(\tau)\right| d\tau$$

so that $s'(t) = |\gamma'(t)|$ is smoothly invertible for *regular* curves (i.e. with $\gamma'(t) \neq 0$).

Then we may parametrisse γ by arclength:

$$\gamma(s) = \gamma(t(s))$$

satisfying $|\gamma'| \equiv 1$.

Area

Let

$$X_u = d\varphi(e_u) = \partial_u \varphi, \quad X_v = d\varphi(e_v) = \partial_v \varphi$$

be coordinate vectors.
Since $d\varphi$ is injective, X_u, X_v form a basis for $T_x M$.
In fact X_u, X_v determines a parallelogram $X_u \wedge X_v \subseteq T_x M$.
Taking a small rectangle $R = \{(u, v) \in (u_0, u_0 + \Delta u) \times (v_0, v_0 + \Delta v)\}$, we approximate the area of $\varphi(R) \subseteq S$ by

$$\operatorname{Area}(\varphi(S)) \simeq \operatorname{Area}(X_u \wedge X_v) = |X_u \times X_v| \operatorname{Area}(R) = |X_u \times X_v| \Delta u \Delta v.$$

Note that $|X_u \times X_v|^2 = \det \lambda^T \lambda = \det g$ where $\lambda = (X_u \quad X_v)$!
Area is the limit of a Riemann sum: for any region $\Omega = \varphi(W) \subseteq \varphi(U)$

$$\operatorname{Area}(\Omega) = \int_W \sqrt{\det g(u, v)} du dv.$$

Intrinsic Geometry

- Notice that thinking of $\gamma : (a, b) \to \mathbb{R}^3$ we have

$$g(\gamma'(0), \gamma'(0)) = \langle \gamma'(0), \gamma'(0) \rangle_{\mathbb{R}^3}$$

  so that the length of tangent vectors and hence the length of curves is precisely the lengths obtained in $\mathbb{R}^3$.
- Similar for angles and for area in terms of X_u, X_v.
- The point is that, if we know g, we may do geometry on S without any reference to how it sits in $\mathbb{R}^3$! This is *intrinsic geometry*.
- But what exactly is the definition of g if we don't refer to $\mathbb{R}^3$?

At this point, the best we can do is say that g is determined by a collection of smooth, matrix valued maps $(u, v) \in U \mapsto (g_{ij}(u, v))$ in each local parametrisation that are symmetric and positive definite at each point (u, v). We also require that under a change of coordinates, τ we have

$$g^{\varphi \circ \tau}_{ab} = \sum_{ij} g^{\varphi}_{ij} \partial_{y^a} \tau^i \partial_{y^b} \tau^j.$$