MATH704 Differential Geometry

Macquarie University, Semester 2 2018

Paul Bryan

Lecture Ten: Differentiable Manifolds

1. Lecture Ten: Differentiable Manifolds
   - Smooth Manifolds
   - Examples
   - Implicit Function Theorem and Regular Values

1. Lecture Ten: Differentiable Manifolds
    - Smooth Manifolds
    - Examples
    - Implicit Function Theorem and Regular Values

Smooth Manifolds: Intrinsic Surfaces

Definition

A set M is an n-dimensional *smooth manifold* if there exists a cover U_α of M and maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ such that

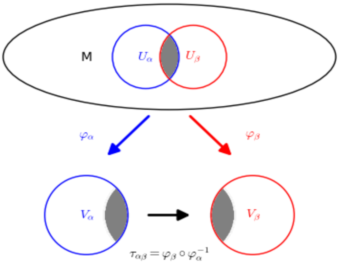
1. each φ_α is a one-to-one and onto an open set $V_\alpha = \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$,
2. $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open,
3. the transition maps

$$\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

   are diffeomorphisms. That is, $\tau_{\alpha\beta}$ is differentiable and has a differentiable inverse.

- In fact, it's enough to assume that $\tau_{\alpha\beta}$ is differentiable for each α, β since $\tau_{\alpha\beta}^{-1} = \tau_{\beta\alpha}$.
- The maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ are called *charts*.
- The collection of all the charts is called an *atlas*.

Charts on a Manifold



Regular Surfaces are Manifolds

Example

Let $S \subset \mathbb{R}^3$ be a regular surface. Then we have a cover of S by local parametrisations

$$\psi_\alpha : V_\alpha \subseteq \mathbb{R}^2 \to \mathbb{R}^3.$$

Then S is a smooth manifold with charts given by

$$\varphi_\alpha = \psi_\alpha^{-1} : U_\alpha = \psi_\alpha(V_\alpha) \subseteq M \to V_\alpha \subseteq \mathbb{R}^2.$$

Note that

$$\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} = \psi_\beta^{-1} \circ \psi_\alpha$$

is differentiable by the assumption that S is a regular surface and by the inverse function theorem.

- Same definition for *regular n-dimensional hypersurfaces* $M^n \subseteq \mathbb{R}^{n+1}$.

Lecture Ten: Differentiable Manifolds - Examples

1. Lecture Ten: Differentiable Manifolds
   - Smooth Manifolds
   - Examples
   - Implicit Function Theorem and Regular Values

Example: The Sphere

Example

The n-sphere is the set

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1\} = \{V \in \mathbb{R}^{n+1} : \|V\| = 1\}.$$

Polar coordinates by induction:

$$\mathbb{S}^n \setminus \{N, S\} = \{(\sqrt{1 - r^2}\sigma, r) : -1 < r < 1, \quad \sigma \in \mathbb{S}^{n-1}\}$$

emispheres:

$$U_1^{\pm} = \{(\pm\sqrt{1 - (x')^2}, x') : x' \in \mathbb{R}^n, \|x'\| < 1\}$$
$$\vdots$$
$$U_{n+1}^{\pm} = \{(x', \pm\sqrt{1 - (x')^2}) : x' \in \mathbb{R}^n, \|x'\| < 1\}$$

Example: The Sphere. Stereographic Coordinates

Example

Draw the line L_x from the North pole $N = e_{n+1}$ to any point $x \in \mathbb{S}^n \setminus \{N\}$.

- That is $L_x = \{(1-t)N + tx : t \in \mathbb{R}\}$.
- Let $\pi_N(x) = (1-t)N + tx : \left\langle e^{n+1}, (1-t)N + tx \right\rangle = 0$. Then $\{\pi_N(x)\} = L_x \cap \{x^{n+1} = 0\}$ is the unique point of intersection of L_x with the $x^{n+1} = 0$ plane.

Then $\pi_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n \simeq \{x^{n+1} = 0\}$ is a bijection.

- In fact
$$\pi_N(x^1, \cdots, x^{n+1}) = \frac{1}{1 - x^{n+1}}(x^1, \cdots, x^n).$$
- The inverse map defined for $y = (y^1, \cdots, y^n)$ is
$$\pi_N^{-1}(y) = \frac{1}{\|y\|^2 + 1}(2y_1, \cdots, 2y_n, \|y\|^2 - 1).$$

Example: The Affine Group.

Example

The affine group is the set of matrices:

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

It corresponds to *orientation preserving affine transformations* $\mathbb{R} \to \mathbb{R}$:

$$x \mapsto ax + b \rightsquigarrow \begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax + b \\ 1 \end{pmatrix}$$

- Smooth manifold with a single chart $\varphi(A_{ij}) = (A_{11}, A_{12})$ maps bijectively with the open set $\{(a, b) \in \mathbb{R}^2 : a > 0\}$.
- Also a regular surface, being the "half space": $\{(a, b, 0) \in \mathbb{R}^3 : a > 0\}$.

Example: Projective Space

Example

Two dimensional real Projective Space, $\mathbb{RP}^2$ is the set of lines through the origin in $\mathbb{R}^3$:

$$\mathbb{RP}^2 = \{[V] : V \neq 0 \in \mathbb{R}^3, \quad [V] = \{\lambda V : V \neq 0\}\}.$$

An atlas is given by three charts. The first:

$$\begin{aligned} \varphi_1 : U_1 = \{[V] = [(V_1, V_2, V_3)] : V_1 \neq 0\} \to \mathbb{R}^2 \\ [V] \mapsto \left(\frac{V_2}{V_1}, \frac{V_3}{V_1}\right). \end{aligned}$$

This maps bijectively with $\mathbb{R}^2$. Similarly U_2 has $V_2 \neq 0$ and U_3 has $V_3 \neq 0$.

Example: Projective Space

Example

- The transition map is defined on $U_1 \cap U_2 = \{[V] : V_1, V_2 \neq 0\}$.
- Then we have

$$\begin{aligned}\varphi_1(U_1 \cap U_2) &= \{\varphi_1([V]) : V_1, V_2 \neq 0\} \\ &= \left\{ \left(\frac{V_2}{V_1}, \frac{V_3}{V_1} \right) : V_1, V_2 \neq 0 \right\} \\ &= \{(x, y) : x \neq 0\}.\end{aligned}$$

- Explicitly

$$\tau_{12} : (x, y) \overset{\varphi_1^{-1}}{\mapsto} [(1, x, y)] \overset{\varphi_2}{\mapsto} (1/x, y/x)$$

- τ_{12} is differentiable for $(x, y) \in \varphi_1(U_1 \cap U_2)$ since then $x \neq 0$.

Example: Grassmanians

Example

The Grassmanian $G_k(\mathbb{R}^n)$ is the set of k-planes $\in \mathbb{R}^n$.
That is,

$$G_k(\mathbb{R}^n) = \{V \subset \mathbb{R}^n | \dim V = k\}.$$

Equivalently,

$$G_k(\mathbb{R}^n) = \{[A : \mathbb{R}^k \to \mathbb{R}^n] | \operatorname{rnk}(A) = k\}$$

where

$$[A] = \{B \cdot A | B : \mathbb{R}^k \to \mathbb{R}^k, \det B \neq 0\}.$$

Lecture Ten: Differentiable Manifolds - Implicit Function Theorem and Regular Values

1. Lecture Ten: Differentiable Manifolds
   - Smooth Manifolds
   - Examples
   - Implicit Function Theorem and Regular Values

Implicit Function Theorem

Theorem (Implicit Function Theorem)

Let $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be a smooth function, let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^k \simeq \mathbb{R}^{n+k}$ and let $z_0 = F(x_0, y_0) \in \mathbb{R}^k$.
If $dF_{(x_0,y_0)}$ restricted to $\{0\} \times \mathbb{R}^k$ is invertible, then there exists an open set U containing x_0 and an open set V containing y_0 and a unique smooth function $g : U \to V$ such that for $(x, y) \in U \times V$ we have $F(x, y) = z_0$ if and only if $y = g(x)$.

Proof.

Let $\bar{F}(x, y) = (x, F(x, y))$. Then $\bar{F} : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ and

$$d\bar{F} = \begin{pmatrix} \mathrm{Id}_n & 0 \\ \partial_x f & \partial_y f \end{pmatrix}$$

where $\partial_x f = df|_{\mathbb{R}^n \times \{0\}}$ and $\partial_y f = df|_{\{0\} \times \mathbb{R}^k}$.

Implicit Function Theorem

Proof.

Then by the inverse function theorem, $\bar{F}$ is locally invertible near (x_0, y_0). Therefore, in a neighbourhood of (x_0, y_0) and a neighbourhood of (x_0, z_0) we have $F(x, y) = z_0$ if and only if $\bar{F}(x, y) = (x, z_0)$ if and only if $(x, y) = \bar{F}^{-1}(x, z_0)$.
Writing $F^{-1}(x, z) = (g_1(x, z), g_2(x, z)) \in \mathbb{R}^{n+k}$, we then take

$$g(x) = g_2(x, z_0).$$

□

- Can you see what the function g_1 must be?
- Notice we proved the implicit function theorem by appealing to the inverse function theorem.
  - In fact the converse is true! Namely, if we assume the implicit function theorem is true, then we can prove the inverse function theorem.
  - But how to prove the implicit function theorem? By the contraction mapping principle of course!

Inverse Image Of A Regular Value

Definition

Let $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be a smooth function. Then $y \in \mathbb{R}^k$ is a *regular value* of F if $\operatorname{rnk} dF_x = k$ for all $x \in F^{-1}(y)$ (i.e. all x such that $F(x) = y$).

Theorem

Let $y \in \mathbb{R}^k$ be a regular value of a smooth function $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$. Then the set $M = F^{-1}(y)$ is a smooth manifold of dimension n.

Proof.

For any $x_0 \in M$, the assumption of the theorem ensures that there are k linearly independent columns in dF_{x_0}.
Label these columns by $i_i, \cdots, i_k$ and label the remaining columns by $j_1, \cdots, j_n$.
By the implicit function theorem, locally near x_0, there is a smooth function $g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^k$ such that $x \in M$ if and only if $(x^{i_1}, \cdots, x^{i_k}) = g(x^{j_1}, \cdots, x^{j_n})$.

Inverse Image Of A Regular Value

Proof.

To make life a little easier, rearrange the columns by permutation so that $(j_1, \cdots, j_n)$ are the first n columns and $(i_1, \cdots, i_k)$ the last k columns. Write π for the permutation that maps x^r to x^{j_r} for $1 \leq r \leq n$ and x^s to x^{i_s} for $1 \leq s \leq k$. This map is a bijection that just rearranges the columns so that the k linearly independent columns are at the end.
Then locally near $\pi^{-1}(x_0)$ we have $\pi(x) \in M$ if and only if $(x^{n+1}, \cdots, x^{n+k}) = g(x^1, \cdots, x^n)$.
Parametrise M near x_0 by

$$\varphi(x^1, \cdots, x^n) \mapsto \pi(x^1, \cdots, x^n, g(x^1, \cdots, x^n)).$$

for $(x^1, \cdots, x^n) \in U$.

Inverse Image Of A Regular Value

Proof.

Then φ is smooth, injective and has injective differential. Thus M is covered by local parametrisations and hence is a regular n-surface in $\mathbb{R}^{n+k}$. Recall for regular surfaces we used the inverse function theorem again to show that the transition maps are diffeomorphisms. The same argument works here and confirms the transition maps are diffeomorphisms, hence M is a manifold. □

- You should try to check the other conditions in the definition of manifold! These, involving only continuity are typically easier.