MATH704 Differential Geometry

Macquarie University, Semester 2 2018

Paul Bryan

Lecture Eleven: The Tangent Space

Lecture Eleven: The Tangent Space - Tangent Vectors

Tangent Vectors

Define an equivalence class of curves: $\gamma \sim \sigma$ if

$$\gamma(0) = \sigma(0)$$

and there is a chart $\varphi : U \to \mathbb{R}^2$ with $\gamma(0) \in U$ such that

$$(\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0).$$

Write $[\gamma] = \{\sigma : \sigma \sim \gamma\}$ for the equivalence class of γ.

Definition

The tangent space, $T_x M$ to M at x is the equivalence class of curves through x

$$T_x M = \{[\gamma] : \gamma(0) = x\}.$$

If we choose a different chart, ψ

$$\begin{aligned}(\psi \circ \gamma)'(0) &= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma)'(0) \\ &= d(\psi \circ \varphi^{-1}) \cdot (\varphi \circ \gamma)'(0) = d(\psi \circ \varphi^{-1}) \cdot (\varphi \circ \sigma)'(0) \\ &= (\psi \circ \sigma)'(0).\end{aligned}$$

Tangent Vectors on Regular Surfaces

Recall that for a regular surface

$$T_x S = \{\gamma'(0) : \gamma(0) = x\}$$

where $\gamma'(0)$ is the derivative at zero of $\gamma : (-\epsilon, \epsilon) \to S \subseteq \mathbb{R}^3$ as a curve in $\mathbb{R}^3$.
The new definition says tangent vectors are equivalence classes of curves $[\gamma]$ in S.
The definitions will be equivalent provided:

- $\gamma'(0) = \sigma'(0)$ as vectors in $\mathbb{R}^3$ if and only if $[\gamma] = [\sigma]$.

Tangent Vectors on Regular Surfaces

Now recall that charts φ are just inverses of local parametrisations ψ.
That is $\varphi = \psi^{-1}$.
We have

$$\gamma'(0) = \sigma'(0) \Leftrightarrow (\varphi^{-1} \circ \varphi \circ \gamma)'(0) = (\varphi^{-1} \circ \varphi \circ \sigma)'(0)$$

if and only if

$$d(\varphi^{-1}) \cdot (\varphi \circ \gamma)'(0) = d(\varphi^{-1}) \cdot (\varphi \circ \sigma)'(0).$$

But $\psi = \varphi^{-1}$ is a local parametrisation so that $d(\varphi^{-1})$ injective.
Therefore the last equation is equivalent to

$$(\varphi \circ \gamma)'(0) = (\varphi \circ \sigma)'(0).$$

That is $[\gamma] = [\sigma]$.

Coordinate Vector Fields

Definition

With respect to chart φ, we define *coordinate vector fields*:

$$E_u(x) = [\varphi^{-1}(\varphi(x) + (t, 0))], \quad E_v(x) = [\varphi^{-1}(\varphi(x) + (0, t))]$$

- That is, $\varphi : U \to \mathbb{R}^2$ and so $\varphi(x) \in \mathbb{R}^2$. Then $\varphi(x) + (t, 0)$ is a curve in $\mathbb{R}^2$ and

  $$\gamma_u(t) = \varphi^{-1}(\varphi(x) + (t, 0))$$

  is a curve in M with $\gamma_u(0) = x$.
- Thus $E_u(x) = [\gamma_u]$ is a tangent vector at x.
- We think of E_u as $\gamma_u'(0)$ (though strictly speaking, the derivative only makes sense in the chart).
- Analogously for n-dimensions: $E_i(x) = [\varphi^{-1}(\varphi(x) + te_i)]$.

Lecture Eleven: The Tangent Space - The Tangent Bundle

Definition

Definition

The set of tangent vectors is called the *tangent bundle*. It is denoted TM.

- Each tangent vector is an equivalence class of curves $X = [\gamma]$.
- There is a *bundle projection* map:

$$x = \pi(X) = \gamma(0) \in M$$

  where $X = [\gamma]$.
- This is independent of the representative since if $X = [\gamma] = [\sigma]$, then by definition $\gamma(0) = \sigma(0)$.

Vector Bundle Structure

Theorem

The tangent bundle is a manifold. In fact, it is a vector bundle of rank $n = \dim(M)$*.*

Definition

A *vector bundle* of *rank* k consists of smooth manifolds M, E and a smooth map $\pi : E \to M$ such that there exists an open cover $\{U_\alpha\}$ of M and *local trivialisations* $\varphi_\alpha : E|_{U_\alpha} := \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ satisfying

1. $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ is a homeomorphism,
2. $p_1 \circ \varphi_\alpha = \pi$ where $p_1 : U_\alpha \times \mathbb{R}^k \to U_\alpha$ is the projection onto the first factor,
3. The *transition maps* $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^k \to U_\alpha \cap U_\beta \times \mathbb{R}^k$ are of the form
$$\tau_{\alpha\beta}(x, V) = (x, A_{\alpha\beta}(x) \cdot V)$$
where $A_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{GL}_n$ is a smooth map with each $A_{\alpha\beta}(x)$ and invertible matrix and $A_{\alpha\beta}(x) \cdot V$ denotes matrix multiplication

Remarks on Vector Bundles

1. $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ is a diffeomorphism,
   This point says that locally a vector bundle is may be identified diffeomorphicly with a *trivial bundle* $U_\alpha \times \mathbb{R}^k$.
2. $p_1 \circ \varphi_\alpha = \pi$ where $p_1 : U_\alpha \times \mathbb{R}^k \to U_\alpha$
   - This just says that under the local indentification with a trivial bundle, the projection is just projecting onto the first factor.
   - For $X \in E$, we have $\varphi_\alpha(X) = (x, V_\alpha)$ with $x \in U_\alpha \subseteq M$ and $V_\alpha \in \mathbb{R}^k$.
   - We think of elements of a vector bundle having a base point $x = \pi(X) = p_1(\varphi_\alpha(X)) = p_1(x, V) = x$ and locally a *vector part* $V \in \mathbb{R}^k$.
3. $$(x, V_\beta) = \tau_{\alpha\beta}(x, V_\alpha) = (x, A_{\alpha\beta}(x) \cdot V_\alpha)$$

   The vector part $V_\alpha = p_2(\varphi_\alpha(X))$ depends on the chosen trivialisation. The transition map tells us how to relate the vector part in one local trivialisation with the vector part in another: $V_\beta = A_{\alpha\beta} \cdot V_\alpha$. Think of this like a *change of basis*.

Fibres and Vector Space Structure

Definition

Let $X_1, X_2 \in TM$ with $x = \pi(X_1) = \pi(X_2)$ and let $c^1, c^2 \in \mathbb{R}$. Then we define

$$c^1 X_1 + c^2 X_2 = \varphi_\alpha^{-1}(x, c^1 V_1^\alpha + c^2 V_2^\alpha)$$

where $\varphi_\alpha(X_i) = (x, V_i^\alpha)$.

In another local trivialisation, we have $(x, V_i^\beta) = (x, A_{\alpha\beta} \cdot V_i^\alpha)$. Then

$$\begin{aligned}
\tau_{\alpha\beta}(\varphi_\alpha(c^1 X_1 + c^2 X_2)) &= (x, A_{\alpha\beta}(x) \cdot (c^1 V_1^\alpha + c^2 V_2^\alpha)) \\
&= (x, c^1 A_{\alpha\beta}(x) X_1^\alpha + c^2 A_{\alpha\beta}(x) X_2^\alpha) \\
&= (x, c^1 V_1^\beta + c^2 V_2^\beta) \\
&= \varphi_\beta(c^1 X^1 + c^2 X^2).
\end{aligned}$$

Thus taking a linear combination of the V_i^α is identified by the transition map with the same linear combination of the V_i^β hence definition of $c_1 X^1 + c_2 X^2$ is independent of the chosen local trivialisation.

Proof of Vector Bundle Structure

Proof.

In a local chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, we have coordinate vector fields

$$E_i(x) = [\varphi_\alpha^{-1}(\varphi_\alpha(x) + te_i)].$$

These are a basis since if $X = [\gamma]$, then

$$(\varphi_\alpha \circ \gamma)'(0) = X^1 e_1 + \cdots + X^n e_n$$

for unique constants $X^1, \ldots, X^n \in \mathbb{R}$.
Therefore

$$\begin{aligned} X &= [\varphi_\alpha^{-1}(\varphi_\alpha(x) + t(X^1 e_1 + \cdots + X^n e_n))] \\ &= X^1[\varphi_\alpha^{-1}(\varphi_\alpha(x) + te_1)] + \cdots + X^n[\varphi_\alpha^{-1}(\varphi_\alpha(x) + te_n)] \\ &= X^1 E_1 + \cdots X^n E_n. \end{aligned}$$

Proof of Vector Bundle Structure

Proof.

For $X \in E|_{U_\alpha} = \pi^{-1}(U_\alpha)$, define

$$\Phi_\alpha(X) = (x, X^1, \ldots, X^n) \in U_\alpha \times \mathbb{R}^k.$$

The first two points in the definition vector bundle are straightforward. For the third, the transition maps are

$$\tau_{\alpha\beta}(x, V) = (x, d(\varphi_\beta \circ \varphi_\alpha^{-1}) \cdot V).$$

Recall that $\varphi_\beta \circ \varphi_\alpha^{-1}$ are the transition maps for M which are smooth diffeomorphisms hence the differential is a *linear isomorphism* as required.

Proof of Vector Bundle Structure

Proof.

For the manifold structure on TM. Charts are given by:

$$\psi_\alpha(X) = (\varphi_\alpha(x), X^1(x), \ldots, X^n(x)) \in \mathbb{R}^n \times \mathbb{R}^n.$$

The transition map is

$$\psi_\beta \circ \psi_\alpha^{-1}(y, V) = (\varphi_\beta \circ \varphi_\alpha^{-1}(y), d(\varphi_\beta \circ \varphi_\alpha^{-1}) \cdot V).$$

□

Examples

1. For $M = \mathbb{R}^n$, we have $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$.
2. On the two-sphere $\mathbb{S}^2$, the famous "Hairy Ball Theorem" from albegraic topology states that there is no non-vanishing vector field on $T\mathbb{S}^2$. Then $T\mathbb{S}^2 \not\simeq \mathbb{S}^2 \times \mathbb{R}^2$.
3. In fact, a much deeper result says that $T\mathbb{S}^n \simeq \mathbb{S}^n \times \mathbb{R}^n$ if and only if $n = 1, 3, 7$.
   - It's not too hard to show the result is true for $n = 1, 3, 7$ by using complex multiplication for $\mathbb{S}^1 \subseteq \mathbb{R}^2 \simeq \mathbb{C}$, and quaternion and octonion multiplication for $n = 3$ and $n = 7$ respectively.
   - The really deep part is that no other n admits a *global trivialisation*.
4. The torus has $T\mathbb{T} \simeq \mathbb{T} \times \mathbb{R}^2$, since $\mathbb{T} \simeq \mathbb{S}^1 \times \mathbb{S}^1$.
5. In general,

$$T(M \times N) \simeq TM \times TN$$

so that the tangent bundle of a product of manifolds is the product of the tangent bundles.

Riemannian Metrics

Definition

A *Riemannian metric* (or just metric) on M is a smooth choice, g_x of positive definite, symmetric bilinear form for each $x \in M$.

There are various ways to interpret the term *smooth* here. In the present context, perhaps the easiest way to define smooth is with respect to the coordinate vector fields: define

$$g_{ij}(x) = g(E_i(x), E_j(x))$$

Then

$$g_x = (g_{ij}(x))$$

is smooth if and only if $y \in \mathbb{R}^n \mapsto (g_{ij}(\varphi^{-1}(y))$ is a smooth matrix valued function. Equivalently, each component function g_{ij} is smooth.

Riemannian Geometry

We can define length, angle and area just as for regular surfaces.

$$|X|_g = \sqrt{g(X, X)}, \quad \text{length of a tangent vector}$$

$$\theta = \arccos\left(\frac{g(X, Y)}{|X|_g |Y|_g}\right) \quad \text{angle between tangent vectors}$$

$$L[\gamma] = \int_a^b |\gamma'(t)| dt \quad \text{arc-length of a curve}$$

$$A(R) = \int_R \sqrt{\det g} \, du dv \quad \text{area of a bounded region}$$