MATH704 Differential Geometry

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Lecture Twelve: Differentiation

1. Lecture Twelve: Differentiation
   - Vector Fields
   - Connections
   - Riemannian (Levi-Civita) Connection

Lecture Twelve: Differentiation - Vector Fields

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Vector Fields

Definition

A *vector field* on a smooth manifold is a smooth function $X : M \to TM$ such that $X(x) \in T_x M$ for each $x \in M$.

- Smoothness means:
  In local coordinates (i.e. in a chart U), we may uniquely write:

$$X(x) = X^1(x)e_1(x) + \cdots + X^n(x)e_n(x)$$

  where $e_1, \cdots, e_n$ are the coordinate vector fields.
  Then X is smooth if the functions $X^i : U \to \mathbb{R}$ are smooth.

Some Examples

Example (On the cylinder)

$$X(x, y, z) = (-y, x, 0), \quad X(z, \theta) = (-\sin\theta, \cos\theta, 0)$$

Example (On the sphere)

$$X(x, y, z) = (1, 0, 0) - \langle (1, 0, 0), (x, y, z) \rangle (x, y, z) = (1 - x^2, -xy, -xz)$$

Example (On a graph, $S = \{(u, v, f(u, v))\}$)

$$X(u, v) = (1, 0, f_u(u, v)), \quad X(u, v) = (0, 1, f_v(u, v))$$

Tangent Vectors as Derivations

Definition

A tangent vector acts as a *local derivation*: For $V \in TM$, with $x = \pi(X)$ and $f : M \to \mathbb{R}$ a smooth function:

$$V(f) := df_x \cdot V = \partial_t|_{t=0} f(\gamma(t))$$

where $V = [\gamma]$.

- Here $V(f) \in \mathbb{R}$ is a real number. In a chart φ:
$$V(f) = d(f \circ \varphi^{-1})|_{\varphi(x)} \cdot (\varphi \circ \gamma)'(0).$$
- Note that f is smooth provided $f \circ \varphi^{-1}$ is smooth for any chart and γ is smooth provided $\varphi \circ \gamma$ is smooth for any chart.
- Notice that
$$d(f \circ \varphi^{-1})|_{\varphi(x)} \cdot (\varphi \circ \gamma)'(0) = \partial_t|_{t=0} \left[(f \circ \varphi^{-1}) \circ (\varphi \circ \gamma) \right] = \partial_t|_{t=0} f(\gamma(t))$$
is independent of the choice of chart.

Vector Fields as Derivations

Definition

Let $X : M \to TM$ be a vector field and $f : M \to \mathbb{R}$ a smooth function. Then we define a new smooth function,

$$X(f)(x) = df_x(X(x)).$$

- Sometimes, we write $X(f)$ as $\partial_X f$ to emphasise that f is differentiated in the direction X.
- In a chart, with $X = X^1 e_1 + \cdots + X^n e_n$ we have

$$(\partial_X f)(\varphi^{-1}(y)) = X^1(y)\frac{\partial f}{\partial y^1}(y) + \cdots X^n(y)\frac{\partial f}{\partial x^n}(y) = D_X f$$

  the usual directional derivative on $\mathbb{R}^n$.
- In particular, if E_i is a coordinate vector field we write ∂_i for E_i since

$$E_i(f) = \frac{\partial f}{\partial x^i}.$$

Leibniz Product Rule

Lemma

Let $f, g : eM \to \mathbb{R}$ be smooth functions. For a tangent vector $V \in TM$ with $x = \pi(V)$, we have

$$V(fg) = f(x)V(g) + g(x)V(f).$$

For a vector field X, we have

$$\partial_X(fg)(x) = f(x)\partial_X g(x) + g(x)\partial_X f(x).$$

- The proof follows from the corresponding rule for the directional derivative in $\mathbb{R}^n$!

Dependence on X and f

Lemma

Let X be a vector field and f be a function. Then at a point $x \in M$, $\partial_X f(x)$ depends on f in a neighbourhood of x (in fact it only on f restricted to γ where $\gamma'(0) = X$) but only on the value $X(x)$ of X at x.

- If f, g are functions such that $f \equiv g$ on an open neighbourhood $U \subseteq M$, then $\partial_X f(x) = \partial_X g(x)$ for every $x \in M$.
- In fact, if γ is any curve with $X = [\gamma]$, then we only need $f \circ \gamma = g \circ \gamma$.
- On the other hand, if X and Y are vector fields such that $X(x) = Y(x)$, then $\partial_X f(x) = \partial_Y f(x)$ even if $X(y) \neq Y(y)$ for every $y \neq x$.

Thus $\partial_X f(x)$ depends on f at nearby points to x but only on X at the point x itself.

The Lie Bracket

Definition

The Lie Bracket $[X, Y]$ of two vector fields X, Y is defined by

$$[X, Y]f = \partial_X f \partial_Y f - \partial_Y \partial_X f.$$

- The point is that although $[X, Y]$ includes second derivatives of f, they all cancel and only first derivatives are left!
- In a chart

$$\partial_X \partial_Y f = \partial_X \left(\sum_{i=1}^n Y^i \partial_i f \right) = \sum_{j=1}^n \sum_{i=1}^n X^j Y^i \partial_j \partial_i f + X^j \partial_j Y^i \partial_i f$$

and

$$\partial_Y \partial_X f = \sum_{i=1}^n \sum_{j=1}^n Y^i X^j \partial_i \partial_j f + Y^i \partial_i X^j \partial_j f.$$

The Lie Bracket

Now we have

$$\begin{aligned}
\partial_X \partial_Y f - \partial_Y \partial_X f &= \sum_{i,j} X^j Y^i \partial_j \partial_i f + X^j \partial_j Y^i \partial_i f \\
&\quad - \sum_{i,j} Y^i X^j \partial_i \partial_j f + Y^i \partial_i X^j \partial_j f \\
&= \sum_{i,j} X^j \partial_j Y^i \partial_i f - \sum_{i,j} Y^i \partial_i X^j \partial_j f \\
&= \sum_{i,j} X^j \partial_j Y^i \partial_i f - \sum_{i,j} Y^j \partial_j X^i \partial_i f \\
&= \sum_i \left[\sum_j X^j \partial_j Y^i - Y^j \partial_j X^i \right] \partial_i f \\
&= \sum_i \left[\partial_X Y^i - \partial_Y X^i \right] \partial_i f.
\end{aligned}$$

The Lie Bracket

That is we have $Z = [X, Y]$ is a vector field expressed in coordinates as

$$Z = Z^i \partial_i$$

with

$$Z^i = \partial_X Y^i - \partial_Y X^i.$$

- The Lie bracket is a *commutator*. It measures the effect of applying Y and then X compared with applying X and then Y.
- It involves derivatives of both X and Y and thus depends on both X and Y in an open neighbourhood.
- Using the Leibniz rule we can verify the Leibniz rule for $[X, Y]$. Exercise!

The Lie Bracket

Example

Locally, let $X = \partial_i$ and $Y = \partial_j$. Then

$$[X, Y] = 0.$$

Example

Let $X = y\partial_x$, $Y = \partial_y$ on $\mathbb{R}^2$. Then

$$[X, Y] = -\partial_x.$$

$\mathrm{h_x} = 0$, $\mathrm{f_y} = 0$

Example

Let $X = f\partial_x + g\partial_y$, $Y = h\partial_x + k\partial_y$ on $\mathbb{R}^2$. Then

$$[X, Y] = (fh_x + gh_y - hf_x - kf_y)\partial_x + (fk_x + gk_y - hg_x - kg_y)\partial_y = (gh_y - hf_x) +$$

Lecture Twelve: Differentiation - Connections

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Directional Derivative

- Let $X, Y : \mathbb{R}^3 \to \mathbb{R}^3$ be vector fields, which we may write uniquely as

$$X(u) = X^x(u)e_x + X^y(u)e_y + X^z(u)e_z, \quad u = (x, y, z) \in \mathbb{R}^3.$$

and similarly for Y.

Definition

The directional derivative, $D_X Y$ is the vector field,

$$\begin{aligned}(D_X Y)(u) = &\Big[X^x(u)\partial_x Y^x(u) + X^y(u)\partial_y Y^x(u) + X^z(u)\partial_z Y^x(u)\Big] e_x \\ &+ \Big[X^x(u)\partial_x Y^y(u) + X^y(u)\partial_y Y^y(u) + X^z(u)\partial_z Y^y(u)\Big] e_y \\ &+ \Big[X^x(u)\partial_x Y^z(u) + X^y(u)\partial_y Y^z(u) + X^z(u)\partial_z Y^z(u)\Big] e_z\end{aligned}$$

- That is, we just differentiate the components:
$D_X Y = (D_X Y^x)e_x + (D_X Y^y)e_y + (D_X Y^z)e_z$.

Directional Derivative on $\mathbb{R}^2$

- Perhaps a more familiar way to write $D_X Y$ is as follows:
- On $\mathbb{R}^2$, write $X = (a, b)$, $Y = (u, v)$. Then

$$D_X Y = \left(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}, a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y} \right).$$

- In terms of the basis $e_x = (1, 0)$, $e_y = (0, 1)$, this is the same as above, just with less components (4 as opposed to 9):

$$D_X Y = (a\partial_x u + b\partial_y u)\, e_x + (a\partial_x v + b\partial_y v)\, e_y.$$

Directional Derivative

- We may also interpret the directional derivative as

$$D_X Y = \partial_t|_{t=0} Y(\gamma(t))$$

where $\gamma'(0) = X$.

- Partial Derivatives

$$\partial_x f(u) = \partial_t|_{t=0} f(u + te_x) = D_{e_x} f,$$

and

$$\partial_x Y = D_{e_x} Y = \partial_x Y^x e_x + \partial_x Y^y e_y + \partial_x Y^x e_z.$$

- We may think of directional derivatives as an operator on smooth functions and vector fields:

$$D_X : f \mapsto D_X f, \quad D_X : Y \mapsto D_X Y$$

Notation for Vector Fields

- Since $\partial_x f = D_{e_x} f$, we write
$$e_x = \partial_x, e_y = \partial_y, e_z = \partial_z.$$
and
$$X = X^x \partial_x + X^y \partial_y + X^z \partial_z$$
Then
$$D_X Y = \sum_{i,j=1}^{3} X^i (\partial_i Y^j) \partial_j$$
where $x_1 = x, x_2 = y, x_3 = z$ and $\partial_i = \partial_{x_i}$.
- Einstein Summation Notation (because writing $\sum$ is too much effort!):
$$D_X Y = X^i \partial_i Y^j \partial_j$$
and anytime there is an upper index repeated as a lower index, there is an implies sum:For example
$$X^i \partial_i = \sum_{i=1}^{3} X^i \partial_i = X^1 \partial_1 + X^2 \partial_2 + X^3 \partial_3.$$

First Attempt at Directional Derivative on a Regular Surface

Definition (First Attempt)

Let S be a regular surface, with $X, Y : S \to \mathbb{R}^3$ tangent vector fields. Define

$$\nabla_X Y = D_X Y.$$

Example (On the Sphere)

Let $X = (1 - x^2, -xy, -xz)$. Then

$$\begin{aligned} D_X X &= \left[(1-x^2)\partial_x - xy\partial_y - xz\partial_z\right]\left[(1-x^2)\partial_x - xy\partial_y - xz\partial_z\right] \\ &= \left[(1-x^2)(-2x)\right]\partial_x + \left[(1-x^2)(-y) - xy(-x)\right]\partial_y \\ &\quad + \left[(1-x^2)(-z) - xz(-x)\right]\partial_z \\ &= (2x^3 - 2x)\partial_x + (2x^2 y - y)\partial_y + (2x^2 z - z)\partial_z. \end{aligned}$$

First Attempt at Directional Derivative on a Regular Surface

Example (On the Sphere (continued))

- We have $D_X X = (2x^3 - 2x)\partial_x + (2x^2 y - y)\partial_y + (2x^2 z - z)\partial_z$.
- Recall $N(u) = (x, y, z) = x\partial_x + y\partial_y + z\partial_z$
- But

$$\begin{aligned}\langle D_X X, N\rangle &= \Big\langle (2x^3 - 2x)\partial_x + (2x^2 y - y)\partial_y + (2x^2 z - z)\partial_z, \\ &\quad x\partial_x + y\partial_y + z\partial_z \Big\rangle \\ &= x(2x^3 - 2x) + y(2x^2 y - y) + z(2x^2 z - z) \\ &= 2x^2(x^2 + y^2 + z^2) - x^2 - (x^2 + y^2 + z^2) \\ &= x^2 - 1.\end{aligned}$$

- Therefore $\langle D_X X(u), N(u)\rangle = x^2 - 1 \not\equiv 0$ and hence $D_X X$ is not tangent in general.

Covariant Derivative

Definition

The covariant derivative $\nabla_X Y$ is defined by

$$\nabla_X Y = D_X Y - \langle D_X Y, N \rangle N$$

That is,

$$\nabla_X Y = \pi_{TS}(D_X Y)$$

is the projection of $D_X Y$ onto the tangent space!
Explicitly, we can see $\nabla_X Y$ is tangential:

$$\langle \nabla_X Y, N \rangle = \langle D_X Y - \langle D_X Y, N \rangle N, N \rangle = \langle D_X Y, N \rangle - \langle D_X Y, N \rangle \langle N, N \rangle = 0$$

since the normal is unit length: $\langle N, N \rangle = 1$.

Covariant Derivative on the Sphere

On the sphere, we simply have

$$\nabla_X Y(u) = D_X Y(u) - \langle D_X Y(u), u \rangle \, u.$$

Example (On the Sphere (revisited))

For $X = (1 - x^2, -xy, -xz)$ we have

$$D_X X = (2x^3 - 2x)\partial_x + (2x^2 y - y)\partial_y + (2x^2 z - z)\partial_z.$$

and

$$\langle D_X X, N \rangle = x^2 - 1.$$

Covariant Derivative on the Sphere

Example (On the Sphere (revisited))

Thus

$$\begin{aligned}
\nabla_X X &= (2x^3 - 2x)\partial_x + (2x^2y - y)\partial_y + (2x^2z - z)\partial_z \\
&\quad - (x^2 - 1)\big[x\partial_x + y\partial_y + z\partial_z\big] \\
&= \quad \big[(2x^3 - 2x) - x(x^2 - 1)\big]\partial_x \\
&\quad + \big[(2x^2y - y) - y(x^2 - 1)\big]\partial_y \\
&\quad + \big[(2x^2z - z) - z(x^2 - 1)\big]\partial_z \\
&= (x^3 - x)\partial_x + x^2y\partial_y + x^2z\partial_z
\end{aligned}$$

Check:

$$\begin{aligned}
\langle \nabla_X Y, N\rangle &= \big\langle (x^3 - x)\partial_x + x^2y\partial_y + x^2z\partial_z, x\partial_x + y\partial_y + z\partial_z \big\rangle \\
&= x^4 - x^2 + x^2y^2 + x^2z^2 \\
&= x^2(x^2 + y^2 + z^2 - 1) = 0.
\end{aligned}$$

Covariant Derivatives on Manifolds

Definition

A *covariant derivative* is a map

$$(X, Y) \mapsto \nabla_X Y$$

with $\nabla_X Y$ a vector field, and such that for functions $f, f^1, f^2 : M \to \mathbb{R}$,

1. Linearity in X: $\nabla_{f^1 X_1 + f^2 X_2} Y = f^1 \nabla_{X^1} Y + f^2 \nabla - X^2 Y$.
2. Additivity in Y: $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$.
3. Product (Leibniz) rule: $\nabla_X (fY) = df(X) Y + f \nabla_X Y$.

- Check directly D is a covariant derivative on $\mathbb{R}^3$.
- On a regular surface (Product Rule. You should check linearity in X!):

$$\begin{aligned}
\nabla_X (fY) &= D_X (fY) - \langle D_X (fY), N \rangle N \\
&= df(X) Y + f D_X Y - \langle df(X) Y + f D_X Y, N \rangle N \\
&= f (D_X Y - \langle D_X Y, N \rangle N) + df(X) Y \\
&= f \nabla_X Y + df(X) Y.
\end{aligned}$$

Coordinate Vector Fields and Christoffel Symbols

In local coordinates $\varphi : U \subseteq S \to V \subseteq \mathbb{R}^2$,

$$X = X^u \partial_u + X^v \partial_v, \quad Y = Y^u \partial_u + Y^v \partial_v.$$

Let $Z = \nabla_X Y$. We want to work out Z^u, Z^v in terms of X^u, X^v, Y^u, Y^v.
Linearity:

$$\nabla_X Y = X^u \nabla_{\partial_u}(Y^u \partial_u) + X^u \nabla_{\partial_u}(Y^v \partial_v) + X^v \nabla_{\partial_v}(Y^u \partial_u) + X^v \nabla_{\partial_v}(Y^v \partial_v)$$

Product rule:

$$\nabla_{\partial_u}(Y^u \partial_u) = (\nabla_u Y^u)\partial_u + Y^u \nabla_u \partial_u$$

Christoffel Symbols. Write $\nabla_u \partial_u$ in terms of ∂_u, ∂_v:

$$\nabla_u \partial_u = \Gamma^u_{uu} \partial_u + \Gamma^v_{uu} \partial_v.$$

$$\nabla_X Y = X^i \nabla_{\partial_i}(Y^j \partial_j) = X^i (\partial_i Y^j)\partial_j + X^i Y^j \Gamma^k_{ij} \partial_k = \left(X^i \partial_i Y^j + X^i Y^k \Gamma^j_{ik} \right) \partial_j$$

Example: Polar Coordinates

Choose local coordinates (r, θ) for the plane:

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta), \quad \phi^{-1}(x,y) = (\sqrt{x^2+y^2}, \arctan(y/x)).$$

$$\partial_r = \cos\theta\partial_x + \sin\theta\partial_y \qquad \partial_x = \frac{x}{\sqrt{x^2+y^2}}\partial_r - \frac{y}{x^2+y^2}\partial_\theta$$

$$\partial_\theta = -r\sin\theta\partial_x + r\cos\theta\partial_y \qquad \partial_y = \frac{y}{\sqrt{x^2+y^2}}\partial_r + \frac{x}{x^2+y^2}\partial_\theta$$

$$\begin{aligned} D_{\partial_r}\partial_r &= D_{\cos\theta\partial_x + \sin\theta\partial_y} \cos\theta\partial_x + \sin\theta\partial_y \\ &= \Big[(\cos\theta\partial_x + \sin\theta\partial_y)\cos\theta\Big]\partial_x + \Big[(\cos\theta\partial_x + \sin\theta\partial_y)\sin\theta\Big]\partial_y \\ &= \Big[\partial_r\cos\theta\Big]\partial_x + \Big[\partial_r\sin\theta\Big]\partial_y = 0. \end{aligned}$$

Therefore

$$\Gamma_{rr}^r = \Gamma_{rr}^\theta = 0.$$

Example: Polar Coordinates

$$
\begin{aligned}
D_{\partial_\theta}\partial_\theta &= D_{\partial_\theta}\Big[- r\sin\theta\partial_x + r\cos\theta\partial_y \Big] \\
&= -\Big[\partial_\theta r\sin\theta\Big]\partial_x + \Big[\partial_\theta r\cos\theta\Big]\partial_y \\
&= -r\cos\theta\partial_x - r\sin\theta\partial_y \\
&= -r\partial_r.
\end{aligned}
$$

Therefore

$$\Gamma^r_{\theta\theta} = -r \quad \Gamma^\theta_{\theta\theta} = 0.$$

Exercise: Calculate

$$
\begin{aligned}
D_{\partial_r}\partial_\theta &= \Gamma^r_{r\theta}\partial_r + \Gamma^\theta_{r\theta}\partial_\theta \\
D_{\partial_\theta}\partial_r &= \Gamma^r_{\theta r}\partial_r + \Gamma^\theta_{\theta r}\partial_\theta
\end{aligned}
$$

Lecture Twelve: Differentiation - Riemannian (Levi-Civita) Connection

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Metric compatability

For a Riemannian manifold (M, g) with X, Y vector fields we have

$$x \mapsto [g(X, Y)](x) := g_x(X(x), Y(x))$$

is a smooth function.
In coordinates,

$$g(X, Y) = g(X^i \partial_i, Y^j \partial_j) = X^i Y^j g(\partial_i, \partial_j) := X^i Y^j g_{ij}.$$

Definition

A connection is *metric compatible* if

$$\partial_X [g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Torsion

- Given two vector fields X, Y, we have a commutator: $\nabla_X Y - \nabla_Y X$.
- It may be that this is non-zero simply because X and Y fail to commute.
- For example, with the Directional derivative on $\mathbb{R}^n$,

$$D_X Y - D_Y X = [X, Y]$$

since $D_X Y = \partial_X Y^i \partial_i$ is just differentiating the components.

Definition

The *torsion tensor* of a connection is

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

A connection is *torsion free* if $T(X, Y) = 0$ for all X, Y.

Fundamental Theorem of Riemannian Geometry

Theorem

Given a Riemannian manifold (M, g), there exists a unique metric compatible, torsion free connection. This connection is referred to as the Levi-Civita connection, or Riemannian connection.

Proof.

$\nabla_X Y$ is uniquely defined by the *Koszul formula*

$$\begin{aligned} 2g(\nabla_X Y, Z) = \partial_X(g(Y, Z)) + \partial_Y(g(X, Z)) - \partial_Z(g(X, Y)) \\ + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y). \end{aligned}$$

□

Fundamental Theorem of Riemannian Geometry

- For fixed X, Y, the right hand side is a linear function of Z, thus there exists a unique vector W such that $g(W, Z) = RHS(Z)$.
- Then we define $\nabla_X Y = \frac{1}{2} W$.
- Then one can check this satisfies the definition of a connection.
- The formula is derived by assuming a metric compatible, torsion free connection exists and showing it must satisfy the Koszul formula which establishes uniqueness.

In coordinates

$$\Gamma_{ij}^k \partial_k := \nabla_{\partial_i} \partial_j = \frac{1}{2} g^{kl} \left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij} \right) \partial_k .$$