MATH704 Differential Geometry

Macquarie University, Semester 2 2018

Paul Bryan

Lecture Thirteen: Curvature and Global Geometry

1. Lecture Thirteen: Curvature and Global Geometry
    - Connection on Regular Surfaces (hypersurfaces, sub-manifolds)
    - Riemannian (Levi-Civita) Connection
    - Second Derivatives
    - Curvature Tensor
    - Gauss Curvature
    - Local Gauss-Bonnet
    - Gauss-Bonnet Theorem (Global)

Lecture Thirteen: Curvature and Global Geometry - Connection on Regular Surfaces (hypersurfaces, sub-manifolds)

Commuting Covariant Derivatives

Lemma

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Proof.

$$\nabla_X Y - \nabla_Y X = D_X Y - A(X, Y) - (D_Y X - A(Y, X)) = D_X Y - D_Y X$$

by symmetry of A. So we only need to verify the lemma for D.

$$\begin{aligned}(D_X Y - D_Y X)f &= \left(D_{X^i\partial_i} Y^j\partial_j - D_{Y^k\partial_k} X^l\partial_l\right) f \\ &= X^i\partial_i(Y^j)\partial_j f + X^i Y^j \partial_i\partial_j f - Y^k\partial_k(X^l)\partial_l f + Y^k X^l \partial_k\partial_l f \\ &= \left(X^i\partial_i(Y^j) - Y^i\partial_i(X^j)\right)\partial_j f \\ &= [X, Y]f.\end{aligned}$$

□

Metric Compatibility

Theorem

The covariant derivative is metric compatible. That is, for all tangent vector fields X, Y, Z

$$\partial_X [g(Y,Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Proof.

First note that the Euclidean directional is metric compatible:
We write

$$\langle X, Y\rangle = \sum_i X^i Y^i = \delta_{ij} X^i Y^j, \quad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\begin{aligned} \partial_X \langle Y, Z\rangle &= D_X(\delta_{ij} Y^i Z^j) = \delta_{ij}(D_X Y^i) Z^j + \delta_{ij} Y^i (D_X Z^j) \\ &= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle . \end{aligned}$$

Metric Compatibility

Proof.

Now the covariant derivative

$$\begin{aligned}
\partial_X g(Y, Z) &= \partial_X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle \\
&= \langle D_X Y - \langle D_X Y, N \rangle N, Z \rangle + \langle Y, D_X Z - \langle D_X Z, N \rangle N \rangle \\
&= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z).
\end{aligned}$$

□

Lecture Thirteen: Curvature and Global Geometry - Riemannian (Levi-Civita) Connection

Metric compatability

For a Riemannian manifold (M, g) with X, Y vector fields we have

$$x \mapsto [g(X, Y)](x) := g_x(X(x), Y(x))$$

is a smooth function.
In coordinates,

$$g(X, Y) = g(X^i \partial_i, Y^j \partial_j) = X^i Y^j g(\partial_i, \partial_j) := X^i Y^j g_{ij}.$$

Definition

A connection is *metric compatible* if

$$\partial_X [g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Torsion

- Given two vector fields X, Y, we have a commutator: $\nabla_X Y - \nabla_Y X$.
- It may be that this is non-zero simply because X and Y fail to commute.
- For example, with the Directional derivative on $\mathbb{R}^n$,

$$D_X Y - D_Y X = [X, Y]$$

since $D_X Y = \partial_X Y^i \partial_i$ is just differentiating the components.

Definition

The *torsion tensor* of a connection is

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

A connection is *torsion free* if $T(X, Y) = 0$ for all X, Y.

Fundamental Theorem of Riemannian Geometry

Theorem

Given a Riemannian manifold (M, g), there exists a unique metric compatible, torsion free connection. This connection is referred to as the Levi-Civita connection, or Riemannian connection.

Proof.

$\nabla_X Y$ is uniquely defined by the *Koszul formula*

$$\begin{aligned} 2g(\nabla_X Y, Z) = &\partial_X(g(Y, Z)) + \partial_Y(g(X, Z)) - \partial_Z(g(X, Y)) \\ &+ g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y). \end{aligned}$$

□

Fundamental Theorem of Riemannian Geometry

- For fixed X, Y, the right hand side is a linear function of Z, thus there exists a unique vector W such that $g(W, Z) = RHS(Z)$.
- Then we define $\nabla_X Y = \frac{1}{2}W$.
- Then one can check this satisfies the definition of a connection.
- The formula is derived by assuming a metric compatible, torsion free connection exists and showing it must satisfy the Koszul formula which establishes uniqueness.

In coordinates

$$\Gamma_{ij}^k \partial_k := \nabla_{\partial_i} \partial_j = \frac{1}{2} g^{kl} \left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij} \right) \partial_k.$$

Lecture Thirteen: Curvature and Global Geometry - Second Derivatives

Differentiating Linear Maps

- Consider a linear map $T : TS \to TS$. For example

$$\mathcal{W} = -dN, \quad \text{or} \quad \nabla X.$$

- For a vector field X, $T(X)$ is a vector field.
- We can differentiate the vector field $T(X)$ to get $\nabla(T(X))$.
- Thus for another vector field Y, we have

$$[\nabla(T(X))](Y) = \nabla_Y(T(X)).$$

- We want to isolate the change in T but X may also be changing and we don't want to include the change of X.
- Thus we define a new linear map, $\nabla_Y T$:

$$(\nabla_Y T)(X) = \nabla_Y(T(X)) - T(\nabla_Y X).$$

Differentiating Linear Maps

Example

On $\mathbb{R}^2$, let

$$M(x) = \begin{pmatrix} xy & \cos(x) \\ 0 & x^2 - y \end{pmatrix}$$

Then

$$D_{\partial_x} M = \begin{pmatrix} y & -\sin(x) \\ 0 & 2x \end{pmatrix}$$

Observe that

$$D_{\partial_x}[M(x)(\partial_x)] = \partial_x \left[\begin{pmatrix} xy & \cos(x) \\ 0 & x^2 - y \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \partial_x \begin{pmatrix} xy \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

$$D_{\partial_x}[M(x)(\partial_x)] = [D_{\partial_x} M(x)](\partial_x) + M(x)(D_{\partial_x}\partial_x) = [D_{\partial_x} M(x)](\partial_x).$$

Differentiating Linear Maps

Example

Take the same M and let $X(x, y) = x\partial_x$.

$$D_{\partial_x} X = \partial_x.$$

$$D_{\partial_x} [M(x)(X)] = \partial_x \left[\begin{pmatrix} xy & \cos(x) \\ 0 & x^2 - y \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \right] = \partial_x \begin{pmatrix} x^2 y \\ 0 \end{pmatrix} = \begin{pmatrix} 2xy \\ 0 \end{pmatrix}$$

$$D_{\partial_x} M = \begin{pmatrix} y & -\sin(x) \\ 0 & 2x \end{pmatrix}$$

$$[D_{\partial_x} M(x)] (X) = xy\partial_x$$

$$M(x)(D_{\partial_x} X) = xy\partial_x$$

Second Covariant Derivative

Definition

Let X be a vector field. The second covariant derivative of X is defined to be the covariant derivative of $T = \nabla X$

$$(\nabla_Y(\nabla X))(Z) = \nabla_Y(\nabla X(Z)) - \nabla X(\nabla_Y Z) = \nabla_Y(\nabla_Z X) - \nabla_{\nabla_Y Z} X.$$

We also write

$$\nabla^2 X(Y, Z) = (\nabla_Y(\nabla X))(Z).$$

or

$$\nabla^2_{Y,Z} X = (\nabla_Y(\nabla X))(Z).$$

Lecture Thirteen: Curvature and Global Geometry - Curvature Tensor

The Curvature Tensor

Definition

The *curvature tensor* is the commutator of second derivatives,

$$\begin{aligned}\mathrm{Rm}(X, Y)Z &= \nabla^2 Z(X, Y) - \nabla^2 Z(Y, X) \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_X Y - \nabla_Y X} Z.\end{aligned}$$

We can write Rm as a commutator

$$\mathrm{Rm}(X, Y) = \nabla^2_{X,Y} - \nabla^2_{Y,X}.$$

- We may writeRm$(X, Y)Z = \nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) - \nabla_{[X,Y]} Z$.
- The $[X, Y]$ term compensates for the fact that $\nabla^2_{X,Y}$ and $\nabla^2_{Y,X}$ might not commute simply because X and Y might not commute.

Symmetries of the Curvature Tensor

Theorem

For any vectors X, Y, Z, W,

1. $\mathrm{Rm}(X, Y)Z = -\mathrm{Rm}(Y, X)Z$,
2. $g(\mathrm{Rm}(X, Y)Z, W) = -g(\mathrm{Rm}(X, Y)W, Z)$,
3. $\mathrm{Rm}(X, Y)Z + \mathrm{Rm}(Y, Z)X + \mathrm{Rm}(Z, X)Y = 0$ *(Bianchi Identity),*

Proof.

1. Anti-symmetry in the first two slots:

$$\begin{aligned}\mathrm{Rm}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= -\nabla_Y \nabla_Y Z + \nabla_Y \nabla_X Z - \nabla_{-[Y,X]} Z \\ &= -\mathrm{Rm}(Y, X)Z.\end{aligned}$$

□

Symmetries of the Curvature Tensor

Proof.

2. Anti-symmetry in the last two slots. This is a little more involved: We use metric compatability.

$$
\begin{aligned}
g(\nabla_X \nabla_Y Z, W) &= \nabla_X g(\nabla_Y Z, W) - g(\nabla_Y Z, \nabla_X W) \\
&= \nabla_X \nabla_Y g(Z, W) - \nabla_X g(Z, \nabla_Y W) \\
&\quad - \nabla_Y g(Z, \nabla_X W) + g(Z, \nabla_Y \nabla_X W).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
g(\nabla_Y \nabla_X Z, W) &= \nabla_Y g(\nabla_X Z, W) - g(\nabla_X Z, \nabla_Y W) \\
&= \nabla_Y \nabla_X g(Z, W) - \nabla_Y g(Z, \nabla_X W) \\
&\quad - \nabla_X g(Z, \nabla_Y W) + g(Z, \nabla_X \nabla_Y W).
\end{aligned}
$$

□

Symmetries of the Curvature Tensor

Proof.

2. Thus,

$$g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X, W) = (\nabla_X\nabla_Y - \nabla_Y\nabla_X)g(Z, W) - g(Z, \nabla_X\nabla_Y W - \nabla_Y\nabla_X W)$$

Subtracting,

$$g(\nabla_{[X,Y]}Z, W) = \nabla_{[X,Y]}g(Z, W) - g(Z, \nabla_{[X,Y]}W)$$

we obtain,

$$\begin{aligned} g(\mathrm{Rm}(X, Y)Z, W) &= g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W) \\ &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})g(Z, W) \\ &\quad - g(Z, \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W) \\ &= \mathrm{Rm}(X, Y)[g(Z, W)] - g(\mathrm{Rm}(X, Y)W, Z). \end{aligned}$$

Symmetries of the Curvature Tensor

Proof.

2. So far we have

$$g(\mathrm{Rm}(X,Y)Z,W) = R(X,Y)[g(Z,W)] - g(\mathrm{Rm}(X,Y)W,Z)$$

where for the smooth function $f = g(Z,W)$ we have

$$\begin{aligned}\mathrm{Rm}(X,Y)f &= \nabla_X\nabla_Y f - \nabla_Y\nabla_X f - \nabla_{[X,Y]}f\\ &= X(Y(f)) - Y(X(f)) - [X,Y](f)\\ &= [X,Y](f) - [X,Y](f) = 0.\end{aligned}$$

Thus we have

$$g(\mathrm{Rm}(X,Y)Z,W) = -g(\mathrm{Rm}(X,Y)W,Z)$$

□

Symmetries of the Curvature Tensor

Proof.

3.

$$\mathrm{Rm}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$
$$\mathrm{Rm}(Y,Z)X = \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X$$
$$\mathrm{Rm}(Z,X)Y = \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y$$

Then we use the symmetry $\nabla_Y Z - \nabla_Z Y = [Y,Z]$:

$$\nabla_X(\nabla_Y Z) = \nabla_X(\nabla_Z Y + [Y,Z]) = \nabla_X\nabla_Z Y + \nabla_X[Y,Z].$$

Notice that the second term in the last of the three lines above contains a term $-\nabla_X\nabla_Z Y$ which cancels with the $\nabla_X\nabla_Z Y$ term here.

□

Symmetries of the Curvature Tensor

Proof.

3. Using the same cancelling for the other terms results in

$$\begin{aligned}
&\mathrm{Rm}(X, Y)Z + \mathrm{Rm}(Y, Z)X + \mathrm{Rm}(Z, X)Y \\
&\quad = \nabla_X[Y, Z] + \nabla_Y[Z, X] + \nabla_Z[X, Y] \\
&\qquad - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y - \nabla_{[X,Y]}Z \\
&\quad = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]].
\end{aligned}$$

again by using symmetry to get the last line.

□

Symmetries of the Curvature Tensor

Proof.

3. To proof is completed by showing the *Jacobi Identity*:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

This is little tedious but is easily computed directly

$$\begin{aligned}
[X,[Y,Z]](f) &= X([Y,Z]f) - [Y,Z](Xf) \\
&= XYZf - XZYf - YZXf + ZYXf \\
[Y,[Z,X]](f) &= YZXf - YXZf - ZXYf + XZYf \\
[Z,[X,Y]](f) &= ZXYf - ZYXf - XYZf + YXZf.
\end{aligned}$$

Summing the three lines everything cancels.

□

Multi-linearity of the Curvature Tensor

- The curvature tensor is a *multi-linear* map. That is, for each fixed Y, Z, W, the map $X \mapsto g(\mathrm{Rm}(X, Y)Z, W)$ is linear. The same goes for the other three slots.
- Thus for example, writing $X = X^u\partial_u + X^v\partial_v$,

$$\begin{aligned} &g(\mathrm{Rm}(X^u\partial_u + X^v\partial_v, Y)Z, W) \\ &\quad = X^u g(\mathrm{Rm}(\partial_u, Y)Z, W) + X^v g(\mathrm{Rm}(\partial_v, Y)Z, W). \end{aligned}$$

- Note: The last two terms in the map $X \mapsto \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ are not linear because of the Leibniz rule. But the extra terms all cancel:

$$\nabla_{fX}\nabla_Y Z = f\nabla_X\nabla_Y Z$$

$$\begin{aligned} \nabla_Y\nabla_{fX}Z - \nabla_{[Y,fX]}Z &= f\nabla_Y\nabla_X Z + Y(f)\nabla_X Z \\ &\quad - f\nabla_{[Y,X]}Z - Y(f)\nabla_X Z \\ &= f(\nabla_Y\nabla_X Z - \nabla_{[Y,X]}Z). \end{aligned}$$

Lecture Thirteen: Curvature and Global Geometry - Gauss Curvature

1. Lecture Thirteen: Curvature and Global Geometry
   - Connection on Regular Surfaces (hypersurfaces, sub-manifolds)
   - Riemannian (Levi-Civita) Connection
   - Second Derivatives
   - Curvature Tensor
   - Gauss Curvature
   - Local Gauss-Bonnet
   - Gauss-Bonnet Theorem (Global)

Curvature Tensor of a Surface

- Multi-linearity means we only need to compute the curvature tensor on basis elements (summation convention!):

$$g(\mathrm{Rm}(X^i\partial_i, Y^j\partial_j), Z^k\partial_k, W^l\partial_l) = X^iY^jZ^kW^l g(\mathrm{Rm}(\partial_i, \partial_j)\partial_k, \partial_l).$$

- In two dimensions, we only have ∂_u, ∂_v.
- Then

$$\mathrm{Rm}(\partial_u, \partial_u)\partial_v = -\mathrm{Rm}(\partial_u, \partial_u)\partial_v$$

hence

$$\mathrm{Rm}(\partial_u, \partial_u)\partial_v = 0.$$

- Applying the other symmetries, we find the only non-zero component is

$$g(\mathrm{Rm}(\partial_u, \partial_v)\partial_u, \partial_v).$$

Curvature Tensor of a Surface

- All other terms can be obtained from the single term. For example,

$$g(\mathrm{Rm}(\partial_u, \partial_v)\partial_u, \partial_v) = -g(\mathrm{Rm}(\partial_v, \partial_u)\partial_u, \partial_v)$$

  etc.
- Thus we may write

$$g(\mathrm{Rm}(\partial_u, \partial_v)\partial_u, \partial_v) = F(\partial_u, \partial_v)$$

  for some scalar valued function.

The Gauss Equation

We can express the curvature tensor in terms of the second fundamental form (i.e. the curvature we already know about).

Lemma

$$g(\mathrm{Rm}(X,Y)X,Y) = -\left(A(X,X)A(Y,Y) - A(X,Y)^2\right) = -\det A(X,Y).$$

Proof.

Recall that we have

$$\nabla_X V = D_X V - \langle D_X V, N\rangle N.$$

Applying this formula to $V = \nabla_Y Z$, we get

$$\begin{aligned} g(\nabla_X\nabla_Y Z, W) &= \langle \nabla_X\nabla_Y Z, W\rangle \\ &= \langle D_X\nabla_Y Z - \langle D_X\nabla_Y Z, N\rangle N, W\rangle \\ &= \langle D_X\nabla_Y Z, W\rangle . \end{aligned}$$

The Gauss Equation

Proof.

From the previous slide:

$$g(\nabla_X \nabla_Y Z, W) = \langle D_X \nabla_Y Z, W \rangle .$$

Computing $D_X \nabla_Y Z$ gives

$$\begin{aligned} D_X \nabla_Y Z &= D_X \left(D_Y Z - \langle D_Y Z, N \rangle N \right) \\ &= D_X D_Y Z - \langle D_X D_Y Z, N \rangle N - \langle D_Y Z, D_X N \rangle N - \langle D_Y Z, N \rangle D_X N. \end{aligned}$$

Since we are taking the inner product with W, we may ignore the middle two (normal) terms.

$$\begin{aligned} g(\nabla_X \nabla_Y Z, W) &= \langle D_X D_Y Z - \langle D_Y Z, N \rangle D_X N, W \rangle \\ &= \langle D_X D_Y Z, W \rangle - A(Y, Z) A(X, W). \end{aligned}$$

The Gauss Equation

Proof.

A similar formula holds for $g(\nabla_Y \nabla_X Z, W)$ so that

$$\begin{aligned} g(\mathrm{Rm}(X, Y)Z, W) &= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W) \\ &= \langle D_X D_Y Z, W\rangle - A(Y, Z)A(X, W) \\ &\quad - \langle D_Y D_X Z, W\rangle + A(X, Z)A(Y, W) \\ &\quad - \left\langle D_{[X,Y]} Z, W \right\rangle \\ &= \langle \mathrm{Rm}_D(X, Y)Z, W\rangle \\ &\quad + A(X, Z)A(Y, W) - A(Y, Z)A(X, W). \end{aligned}$$

But $\mathrm{Rm}_D \equiv 0$ (Euclidean space has zero curvature!) since

$$D_{\partial_i} D_{\partial_j} \partial_k = 0.$$

□

Covariant Derivative and Curvature are Intrinsic

- We won't prove the theorem here (though it's not difficult).
- The theorem says that we may write $\nabla = \nabla(g)$. That is ∇ may be obtained in a unique way from g.
- Thus we obtain:

Corollary

If (S_1, g_1) and (S_2, g_2) are isometric, then $\nabla_1 = \nabla_2$.

Corollary

If (S_1, g_1) and (S_2, g_2) are isometric, then $\mathrm{Rm}_1 = \mathrm{Rm}_2$.

Gauss' Theorema Egregium (Remarkable Theorem)

Theorem

The Gauss curvature is intrinsic. That is, if (S_1, g_1) and (S_2, g_2) are locally isometric, then $K_1 = K_2$.

Proof.

For any X, Y linearly independent,

$$K = \frac{\det A(X,Y)}{\det g(X,Y)} = -\frac{g(\mathrm{Rm}(X,Y)X,Y)}{\det g(X,Y)}.$$

That's it! The curvature tensor is intrinsic $\mathrm{Rm} = \mathrm{Rm}(\nabla) = \mathrm{Rm}(\nabla(g))$. $\square$

Non-isometric Surfaces

Example

The surfaces

- Sphere: $K \equiv 1$
- Torus: K non-constant but changing sign
- Cylinder: $K \equiv 0$
- Paraboloid: K non-constant and positive

are not locally isometric.

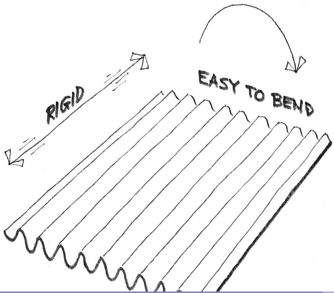
- Besides the cylinder, none of these surfaces can be flattened out (even locally!) without distorting the geometry - stretching, crumpling etc.
- In particular, all surfaces are locally diffeomorphic to the plane (via the local parametrisations) so they share the Calculus with the plane.
- But typically, they do not share the *Geometry* with the plane.

Even though plane calculus may be brought to bear on the study of surface geometry, the geometry itself is not plane geometry.

Corrugation

Example

- Folding a sheet of (paper, metal, cardboard) along a line introduces curvature but does not change the geometry provided no stretching occurs.
- Thus one principal curvature is non-zero, but Gauss' theorem forces the other to vanish since $0 \underset{\text{Gauss Theorem}}{=} K = \kappa_1 \kappa_2$.
- Introduces rigidity in one direction and flexibility in the other.

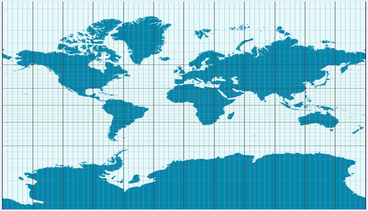


Map Making

Example

- No map exists preserving length, angle and area!
- Archimedes Cylinder to Sphere map preserves area: $(x, y, z) \in C \mapsto (\sqrt{1-z^2}x, \sqrt{1-z^2}y, z)$.
- The Mercator projection preserves angles. Good for navigation!

Pictures



Helicoid and Catenoid

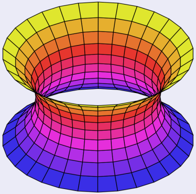
Example

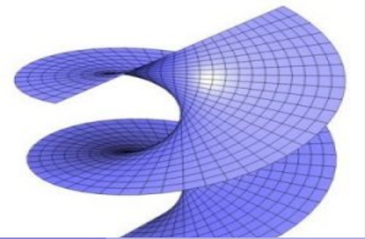
- Helicoid: $(v\cos(u), v\sin(u), u)$,
- Catenoid: $(\sinh(v)\cos(u), \sinh(v)\sin(u), u)$.

The Helicoid and Catenoid are locally isometric with Gauss curvature

$$K = -\frac{1}{(1+u^2)^2}$$

Pictures





The Converse of Gauss' Theorem is false

Example

Here is an example of surfaces S_1, S_2 for which $K_1 = K_2$ but $g_1 \neq g_2$.

- $\varphi(u, v) = (u\cos(v), u\sin(v), \ln(u))$
- $\psi(u, v) = (u\cos(v), u\sin(v), v)$

Exercise:

- Check that $K_\varphi(u, v) = K_\psi(u, v)$
- Check that $g_\varphi(u, v) \neq g_\psi(u, v)$.

- Thus we have surfaces with the equal Gauss curvature that are not isometric.
- Gauss Theorem: $g_1 = g_2 \Rightarrow K_1 = K_2$.
- Converse is false: $K_1 = K_2 \not\Rightarrow g_1 = g_2$.

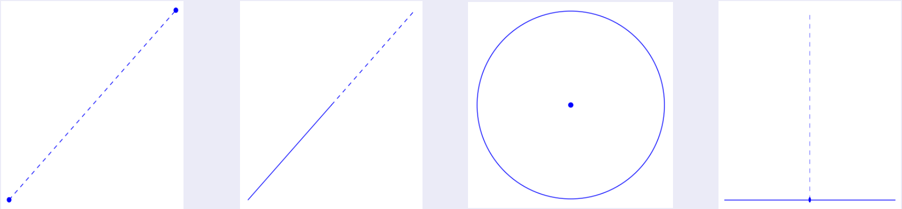
Lecture Thirteen: Curvature and Global Geometry - Local Gauss-Bonnet

Euclid's Axioms for Geometry:

The development of Riemannian geometry began with investigations into whether non-Euclidean geometries exist. Euclidean axioms:

1. Through any two points lies a line,
2. Any (finite) line may be extended indefinitely and uniquely a a straight line
3. Through any point and given any positive number, there exists a circle centred on the point with radius the given number
4. Through any point on a line, there is a unique perpendicular line.

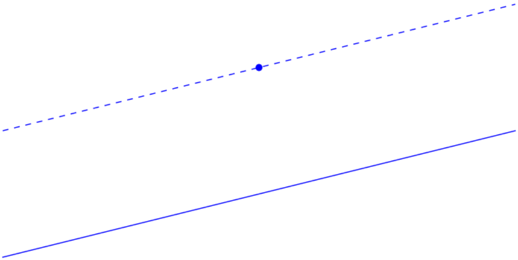
Euclid's First Four Axioms



Parallel Postulate

The first four axioms (or postulates) are relatively self evident and non-controversial.
Of a rather different nature is the famous *fifth postulate*:

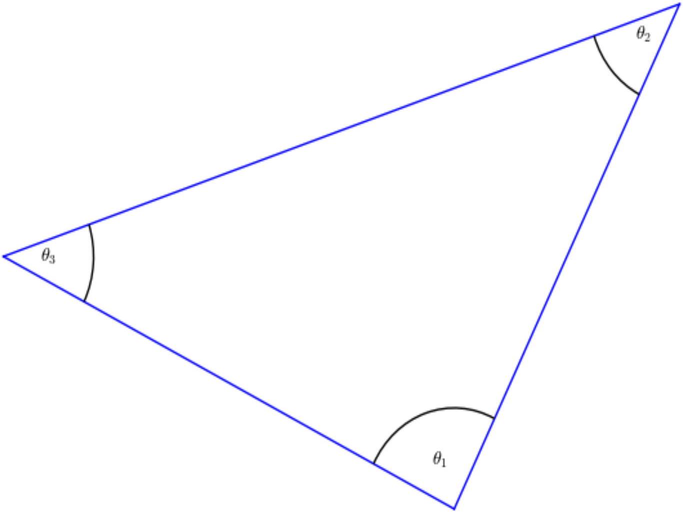
5. Given any line and point not on the line, there *exists a unique* line through the point not intersecting the original line.



Parallel Postulate and Triangles

The fifth postulate is equivalent to:

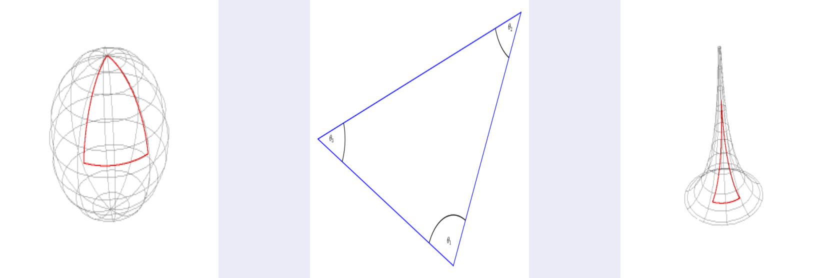
- Sum of the interior angles of a triangle: $\theta_1 + \theta_2 + \theta_3 = \pi$.



Triangles in non-Euclidean Geometry

- Sphere $K > 0$: $\theta_1 + \theta_2 + \theta_3 > \pi$
- Euclidean Space $K = 0$: $\theta_1 + \theta_2 + \theta_3 = \pi$
- Pseudosphere $K < 0$: $\theta_1 + \theta_2 + \theta_3 < \pi$

Constant Curvature Geometries



Piecewise Regular Curves

Definition

A *piecewise* regular curve $\gamma : [a, b] \to S$ is a *continuous* curve such that there exists a partition

$$a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$$

with γ is regular on $[t_i, t_{i+1}]$. The points $\gamma(t_i)$ are called the vertices.

Regular means differentiable and $\gamma' \neq 0$ with left and right continuous limits: $\lim_{t \to^+ t_i} \gamma'(t)$ and $\lim_{t \to^- t_i} \gamma'(t)$ are defined and non-zero.
We write

$$\gamma'_-(t_i) = \lim_{t \to^- t_i} \gamma'(t), \quad \gamma'_+(t_i) = \lim_{t \to^+ t_i} \gamma'(t).$$

Simple Closed Curves

Definition

A *closed curve* is a continuous curve $\gamma : [a, b] \to S$ with $\gamma(a) = \gamma(b)$. A *simple curve* is a curve with no self intersections: $\gamma(t) = \gamma(r) \Rightarrow t = r$.

We consider piecewise regular, simple, closed curves.

Turning Tangents and Total Curvature of Plane Curves

Our first *Global* result for curves. Generalising this result will lead us to the Gauss-Bonnet theorem.

Theorem (Turning Tangents)

Let $\gamma : [0, L] \to \mathbb{R}^2$ *closed plane curve parametrised by arc-length. Then*

$$I := \frac{1}{2\pi} \int_0^L \kappa(s) ds \in \mathbb{Z}.$$

The integer, I is called the winding number. In particular, if γ is a simple, closed curve then

$$\int_0^L \kappa(s) ds = \pm 2\pi.$$

The sign $\pm$ is just the orientation.

Turning Tangents and Total Curvature of Plane Curves

Proof.

- The function

$$\theta(s) = \int_0^s \kappa(\tilde{s})d\tilde{s}$$

  satisfies

$$\partial_s \theta = \kappa.$$

- Since $\partial_s \theta = \kappa$ we have

$$\theta(L) - \theta(0) = \int_0^L \kappa(s)ds.$$

Turning Tangents and Total Curvature of Plane Curves

Proof.

- On the other hand, since $T(s) = \gamma'(s)$ is unit length,

$$T(s) = (\cos(\varphi(s)), \sin(\varphi(s)))$$

  for a differentiable (by the implicit function theorem) function $\varphi : [0, L] \to \mathbb{R}$.
- But $T(L) = T(0)$ and hence

$$\varphi(L) = \varphi(0) + 2\pi I$$

  for an integer I.
- We also have

$$\kappa = \langle \partial_s T, N \rangle = \langle \partial_s \varphi(-\sin\varphi, \cos\varphi), (-\sin\varphi, \cos\varphi) \rangle = \partial_s \varphi.$$

Turning Tangents and Total Curvature of Plane Curves

Proof.

- We have
$$\partial_s \varphi = \partial_s \theta \Rightarrow \varphi(s) = \theta(s) + C$$
for some constant C.
- Therefore,
$$\varphi(L) - \varphi(0) = (\theta(L) + C) - (\theta(0) + C) = \theta(L) - \theta(0)$$
- Putting it all together, we have
$$2\pi I = \varphi(L) - \varphi(0) = \theta(L) - \theta(0) = \int_0^L \kappa ds.$$
- Note θ is just the angle of T with a fixed vector (such as the x-axis).

□

Angle in General

Define the angle θ_i between $\gamma'_-(t_i)$ and $\gamma'_+(t_1)$ as follows:

1. $$|\theta| = \left|\arccos g(T_i^-, T_i^+)\right| \in (0, \pi).$$

   where $T = \gamma' / |\gamma'|$ is the unit tangent.
2. We take $\theta \in (-\pi, \pi)$ by choosing the sign so that $\theta > 0$ whenever

   $$\{T_i^-, T_i^+\}$$

   is positively oriented and $\theta < 0$ otherwise.
3. The case of a *cusp* is when $\theta = \pi$ in which case it's possibly to choose the sign so that θ varies continuously.

Gauss-Bonnet Theorem (Local)

Theorem

Let $D \subseteq S$ be homeomorphic to a disc with boundary a piecewise regular, simple, closed curve, γ. Then

$$\int_D K dA + \int_\gamma \kappa ds = 2\pi - \sum_{i=1}^{k} \theta_i.$$

- Since γ is only piecewise regular, the curvature is not defined at the vertices t_i so we make the definition,

$$\int_\gamma \kappa ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \kappa ds.$$

Proof in the Plane

- In the plane $K \equiv 0$ so Gauss-Bonnet becomes

$$\int_\gamma \kappa ds = 2\pi - \sum_{i=1}^{k} \theta_i.$$

- For γ regular (no vertices) Turning Tangents gives

$$\int_\gamma \kappa ds = 2\pi.$$

- For piecewise regular, break up the integral at the vertices:

$$\begin{aligned}\int_\gamma \kappa ds = \sum \int_{t_{i-1}}^{t_i} \kappa ds = \sum \int_{t_{i-1}}^{t_i} \partial_s \theta ds &= \sum \theta^-(t_i) - \theta^+(t_{i-1}) \\ &= \theta(t_k)^- - \theta(t_0)^+ + \sum \theta^-(t_i) - \theta^+(t_i) \\ &= 2\pi - \sum \theta_i.\end{aligned}$$

Proof of Gauss-Bonnet

Proof.

[sketch in the case D is contained in a local parametrisation]

- On a surface, we may change coordinates so that

$$g = \begin{pmatrix} g_{uu} & 0 \\ 0 & g_{vv} \end{pmatrix}.$$

- The geodesic curvature of $\gamma(s) = (u(s), v(s))$ may be expressed as

$$\kappa = \frac{1}{2\sqrt{g_{uu}g_{vv}}} \left(\partial_v g_{vv} \partial_s v - \partial_u g_{uu} \partial_s u \right) + \partial_s \theta.$$

  Note: In the plane, $g_{uu} = g_{vv} = 1$ and so the first term vanishes recovering the plane case.

Proof of Gauss-Bonnet

Proof.

- Integrating the geodesic curvature,

$$\begin{aligned}\int_{t_{i-1}}^{t_i} \kappa ds &= \int_{t_{i-1}}^{t_i} \frac{1}{2\sqrt{g_{uu}g_{vv}}} \left(\partial_v g_{vv} \partial_s v - \partial_u g_{uu} \partial_s u\right) ds + \int_{t_{i-1}}^{t_i} \partial_s \theta ds \\ &= \int_{t_{i-1}}^{t_i} \left(\frac{1}{2\sqrt{g_{uu}g_{vv}}} \partial_v g_{vv}\right) \partial_s v - \left(\frac{1}{2\sqrt{g_{uu}g_{vv}}} \partial_u g_{uu}\right) \partial_s u ds \\ &\quad + \theta(t_i) - \theta(t_{i-1})\end{aligned}$$

Proof of Gauss-Bonnet

Proof.

Apply the Gauss-Green Theorem:

$$\int_{\gamma} P\partial_u s + Q\partial_v s ds = \int_D \partial_u Q - \partial_v P dA$$

to

$$\int_{\gamma} \kappa ds = \sum \int_{t_{i-1}}^{t_i} \kappa ds$$
$$= \int_{t_{i-1}}^{t_i} \left(\frac{1}{2\sqrt{g_{uu}g_{vv}}}\partial_v g_{vv}\right)\partial_s v - \left(\frac{1}{2\sqrt{g_{uu}g_{vv}}}\partial_u g_{uu}\right)\partial_s u ds$$
$$+ \sum \theta(t_i) - \theta(t_{i-1})$$

Proof of Gauss-Bonnet

Proof.

By Gauss-Green with

$$P = -\frac{1}{2\sqrt{g_{uu}g_{vv}}}\partial_u g_{uu}, \quad Q = \frac{1}{2\sqrt{g_{uu}g_{vv}}}\partial_v g_{vv}$$

we get

$$\int_\gamma \kappa ds = \int_D \partial_u \left(\frac{1}{2\sqrt{g_{uu}g_{vv}}}\partial_v g_{vv}\right) + \partial_v \left(\frac{1}{2\sqrt{g_{uu}g_{vv}}}\partial_u g_{uu}\right) dA$$
$$+ \sum \theta(t_i) - \theta(t_{i-1})$$

Proof of Gauss-Bonnet

Proof.

In our coordinate system with ($g_{uv} = g_{vu} = 0$) the integrand just so happens to be the Gauss curvature:

$$K = \partial_u \left(\frac{1}{2\sqrt{g_{uu} g_{vv}}} \partial_v g_{vv} \right) + \partial_v \left(\frac{1}{2\sqrt{g_{uu} g_{vv}}} \partial_u g_{uu} \right)$$

Thus

$$\int_\gamma \kappa ds = \int_D K + \sum \theta(t_i) - \theta(t_{i-1}) = \int_D K + 2\pi - \sum \theta_i$$

as required. □

Remarks

- The desired coordinate system ($g_{uv} = 0$) is called orthogonal and exists on surfaces locally
- We used a form of the Turning Tangents theorem without proof.
- The formula for κ can be obtained by a similar manner to the plane case $\partial_s \theta = \kappa$ but taking into account the changing metric.
- The formula for K can be obtained from expressing Rm in terms of g and using the Gauss equation.
- The entire proof may be re-written (in a coordinate free way) using the language of *differential forms* where the Gauss-Green theorem appears as Stokes' theorem for differential forms.

Triangles Again

Definition

A *geodesic triangle* is a piecewise regular, simple closed curve with precisely three vertices that is the boundary of a region D homeomorphic to a disc and such that each regular arc is a geodesic.

Let $\varphi_i = \pi - \theta_i \in (0, 2\pi)$ be the *interior angles*. Then

$$2\pi - (\theta_1 + \theta_2 + \theta_3) = 2\pi - (\pi - \varphi_1 + \pi - \varphi_2 + \pi - \varphi_3) = \varphi_1 + \varphi_2 + \varphi_3 - \pi.$$

By Gauss-Bonnet

$$\int_D K dA = 2\pi - (\theta_1 + \theta_2 + \theta_3) = \varphi_1 + \varphi_2 + \varphi_3 - \pi.$$

Triangles in Constant Curvature

Example

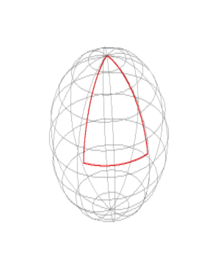
- Sphere $K \equiv 1$: $0 < \text{Area}(D) = \int_D K dA = \varphi_1 + \varphi_2 + \varphi_3 - \pi$.
- Plane $K \equiv 0$: $0 = \int_D K dA = \varphi_1 + \varphi_2 + \varphi_3 - \pi$.
- Pseudosphere $K \equiv -1$:
  $0 > -\text{Area}(D) = \int_D K dA = \varphi_1 + \varphi_2 + \varphi_3 - \pi$.

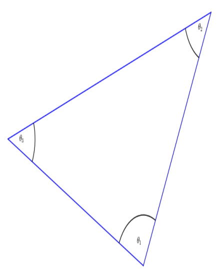
- On the sphere and pseudosphere, the angles determine the area of the triangle!
- On the plane, congruent triangles have the same angles but not generally the same area.

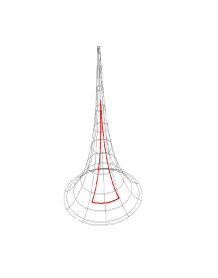
Triangles in non-Euclidean Geometry

- Sphere $K > 0$: $\varphi_1 + \varphi_2 + \varphi_3 = \text{Area}(D) + \pi > \pi$
- Euclidean Space $K = 0$: $\varphi_1 + \varphi_2 + \varphi_3 = \pi$
- Pseudosphere $K < 0$: $\varphi_1 + \varphi_2 + \varphi_3 = -\text{Area}(D) + \pi < \pi$

Constant Curvature Geometries







Regular Tilings

Definition

A *regular n-gon* of S is a piecewise regular, simple, closed curved with n vertices, bounding a disc whose arcs are all geodesics of the same length meeting at the same angle θ.

Let P_i denote a regular n-gon including the boundary curve and the interior.

Definition

A *regular tiling* of S is a set of regular n-gons P_i all of the same area such that

1. $S = \cup_i P_i$
2. For $i \neq j$, $P_i \cap P_j$ is either empty, a vertex, or an entire arc.

Planar Regular Tilings

- In the plane, the interior angle of a regular n-gon is

$$\theta = \pi - 2\pi/n.$$

- Let k be the number of n-gons meeting at a vertex so that adding k copies of θ gives 2π:

$$2\pi = k\theta = k(\pi - 2\pi/n) = \frac{kn - 2k}{n}\pi$$

- Therefore

$$2n = kn - 2k$$

- That is

$$0 = kn - 2k - 2n = k(n-2) - 2(n-2) - 4 = (k-2)(n-2) - 4$$

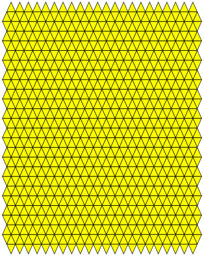
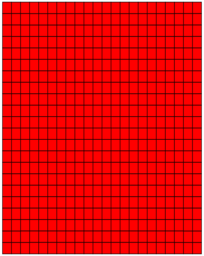
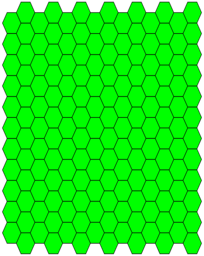
Planar Regular Tilings

- The only solutions (k, n) to

$$(k-2)(n-2) = 4$$

are

$$(k, n) = (6, 3), (4, 4), (3, 6).$$

Spherical Regular Tilings

Example

On the sphere:

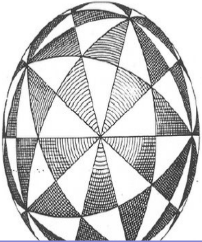
$$2\pi > \frac{kn - 2k}{n}\pi.$$

Hence

$$(k-2)(n-2) < 4$$

Not many solutions...

- *Congruent* but not regular polygons allows more possibilities:

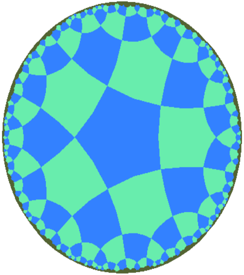


Hyperbolic Tiling

The Poincaré disc is the unit disc $D = \{x^2 + y^2 < 1\}$ equipped with a metric g such that $K \equiv -1$. Gauss-Bonnet applies.

Example

- Now we have
$$(k-2)(n-2) > 4$$
Infinitely many solutions!



Lecture Thirteen: Curvature and Global Geometry - Gauss-Bonnet Theorem (Global)

Triangulations

Definition

A *triangulation* of a regular surface S is a finite set of triangles, $\{T_i\}_{i=1}^n$ such that

1. $S = \cup_{i=1}^n T_i$,
2. Each intersection $T_i \cap T_j$ is either empty, a common edge of T_i and T_j or a common vertex of T_i and T_j.

A fundamental fact we use (without proof) is that there always exists triangulations of surfaces.
Let

$$\begin{aligned} F &= \text{number of triangles (faces)} \\ E &= \text{number of edges} \\ V &= \text{number of vertices.} \end{aligned}$$

Euler Characteristic

Definition

The Euler characteristic, χ of $\{T_i\}_{i=1}^n$ is defined be

$$\chi = V - E + F.$$

Theorem (without proof)

The Euler characteristic is independent of the choice of triangulation. Thus we may define the Euler characteristic of a surface, $\chi(S)$ to be equal to the (common) Euler characteristic of any triangulation.

The Euler characteristic is a *complete topological invariant* for compact surfaces S_1, S_2:

Theorem (without proof)

If $\varphi : S_1 \to S_1$ is a homeomorphism, then $\chi(S_1) = \chi(S_1)$. Conversely, if $\chi(S_1) = \chi(S_2)$, then there exists a homeomorphism $S_1 \to S_2$.

Examples

- disc
- square
- annulis

Examples

- sphere
- torus
- g handles

Classification of Closed Surfaces (compact, no boundary)

Definition

A genus $g \in \mathbb{N} = \{0, 1, 2, \dots\}$ surface S_g is homeomorphic to a sphere with g handles attached.

For every $g \in \mathbb{N}$, there exists such a surface.

Theorem (without proof)

1. $\chi(S_g) = 2(1 - g)$
2. *Every compact surface has* $\chi(S) \in \{-2, 0, -2, -4, \dots, -2k, \dots\}$.

Therefore every compact surface is homeomorphic to S_g *for some* g.

The proof follows by first showing that $\chi(\mathbb{S}^2) = 2$, and then $\chi(S + \text{handle}) = \chi(S) - 2$.

Classification of Closed Surfaces

- Some pictures of genus g surfaces.

Global Gauss-Bonnet

Let $R \subseteq S$ be a *regular region*. That is, R is a region bounded by finitely many piecewise regular, simple, closed curves $\{C_i\}_{i=1}^k$.

Theorem (Global Gauss-Bonnet)

$$\int_R K dA + \sum_{i=1}^k \left(\int_{C_i} \kappa ds + \sum_{j=1}^{N_i} \theta_{ij} \right) = 2\pi\chi(R).$$

- We define

$$\int_R K dA = \sum_n \int_{T_n} K du dv$$

  where $\{T_n\}$ is a triangulation of R with each triangle contained in a local parametrisation.
- For each i, $\{\theta_{ij}\}_{j=1}^{N_i}$ denotes the exterior angles of C_i at the vertices.

Global Gauss-Bonnet Corollaries

Corollary

Let S be a compact, orientable, regular surface. Then

$$\int_S KdA = 2\pi\chi(S).$$

- This is quite an amazing result! Compare all the possible *topological* sphere with widely varying geometry. No matter what, the Gauss curvature distributes itself in such a way that the total Gauss curvature K (i.e. $\int_S KdA$) is the same.

Global Gauss-Bonnet Corollaries

- The standard torus and coffee cup are homeomorphic hence have the same total Gauss curvature.
- A g holed torus and the sphere with g handles attached are homeomorphic, hence have the same total Gauss curvature.
- The Gauss-Bonnet theorem holds also for compact two-dimensional Riemannian manifolds without boundary (closed Riemannian surface). In each homeomorphism class (all surfaces with the same Euler characterstic), there exists a unique (up to scale) closed Riemannian surface, M with constant Gauss curvature given by

$$K \equiv \frac{2\pi\chi(M)}{\text{Area}(M)}.$$

Global Gauss-Bonnet Corollaries

Corollary

Any compact, regular surface, S with $K > 0$ is homeomorphic to the sphere.

Proof.

Gauss-Bonnet implies

$$\chi(S) = \int K dA > 0$$

and hence $\chi(S) = 2$, hence S is homeomorphic to the sphere since χ is a complete invariant. □

Global Gauss-Bonnet Corollaries

- In fact, every compact, regular surface S has an elliptic point (a point where $K > 0$).
- This follows in a similar manner to the proof of the surjectivity of the Gauss map, but rather than taking a plane and moving it until it touches S, one takes a sphere containing S and shrinks it until it touches S. The second derivative test applied to the same function as in the Gauss map proof shows $K > 0$.

Corollary

Every compact, regular surface with $\chi \leq 0$ has points of positive and negative Gauss curvature.

Theorem (A variant of Hilbert's Theorem)

There are no compact, regular surfaces with everywhere negative Gauss curvature.

Proof of Global Gauss-Bonnet Theorem

- Applying the local Gauss-Bonnet Theorem to each triangle T_n with boundary arcs $\gamma_n^1, \gamma_n^2, \gamma_n^3$ in a triangulation,

$$\int_{T_n} K dA + \sum_{m=1}^{3} \left(\int_{\gamma_n^m} \kappa ds + \alpha_{nm} \right) = 2\pi.$$

  where $\alpha_{j1}, \alpha_{j2}, \alpha_{j3}$ are the external angles of the triangle T_j.
- Summing over the number F of triangles, all *interior* arcs appear exactly twice with opposite orientation hence cancel and all that is left are the boundary arcs C_i (see figure). Therefore,

$$\int_R K dA + \sum_{i=1}^{k} \int_{C_i} \kappa ds + \sum_{n=1}^{F} \sum_{m=1}^{3} \alpha_{nm} = 2\pi F.$$

Proof of Global Gauss-Bonnet Theorem

- We have

$$\int_R K dA + \sum_{i=1}^{k} \int_{C_i} \kappa ds + \sum_{n=1}^{F} \sum_{m=1}^{3} \alpha_{nm} = 2\pi F.$$

- Recall the theorem states that

$$\int_R K dA + \sum_{i=1}^{k} \left(\int_{C_i} \kappa ds + \sum_{j=1}^{N_i} \theta_{ij} \right) = 2\pi\chi(R) = 2\pi(F - E + V).$$

- Thus to prove the theorem we need to prove that

$$\sum_{n=1}^{F} \sum_{m=1}^{3} \alpha_{nm} = \sum_{i=1}^{k} \sum_{j=1}^{N_i} \theta_{ij} + 2\pi(E - V)$$

Proof of Global Gauss-Bonnet Theorem

- Let $\beta_{nm} = \pi - \alpha_{nm}$ be the *internal* angles of the triangle T_n.
- Recall the sum is over $1 \leq n \leq F$ and $1 \leq m \leq 3$.
- Then
$$\sum \alpha_{nm} = \sum \pi - \beta_{nm} = 3\pi F - \sum \beta_{nm}.$$
- Thus we now want to show that
$$3\pi F - \sum \beta_{nm} = \sum \theta_{ij} + 2\pi(E - V)$$

Proof of Global Gauss-Bonnet Theorem

- The idea is now to keep track of the edges that lie on a boundary curve C_i (*exterior edges*) and those that lie in the interior of R (*interior edges*).
- Thus we define

$$\begin{aligned} E_{\text{ext}} &= \text{number of exterior edges} \\ E_{\text{int}} &= \text{number of interior edges} \\ V_{\text{ext}} &= \text{number of exterior vertices} \\ V_{\text{int}} &= \text{number of interior vertices} \end{aligned}$$

Proof of Global Gauss-Bonnet Theorem

- Because the C_i are simple, closed curves, we have $V_{\text{ext}} = E_{\text{ext}}$.
- By induction on the number of triangles: $3F = 2E_{\text{int}} + E_{\text{ext}}$.
- Thus we have

$$\begin{aligned} 3\pi F - \sum \beta_{nm} &= 2\pi E_{\text{int}} + \pi E_{\text{ext}} - \sum \beta_{nm} + 2\pi E_{\text{ext}} - 2\pi V_{\text{ext}} \\ &= 2\pi E_{\text{int}} + 2\pi E_{\text{ext}} + \pi E_{\text{ext}} - 2\pi V_{\text{ext}} - \sum \beta_{nm} \\ &= 2\pi E - \pi V_{\text{ext}} - \sum \beta_{nm}. \end{aligned}$$

- To finally finish we need to show that

$$-\pi V_{\text{ext}} - \sum \beta_{nm} = -2\pi V + \sum \theta_{ij}.$$

Proof of Global Gauss-Bonnet Theorem

- Divide the β_{nm} into internal and external vertices

$$\sum \beta_{mn} = \sum_a \beta_{\text{int},a} + \sum_b \beta_{\text{ext},b}$$

- For the internal vertices, the sum of the angles equals to 2π, hence

$$\sum_a \beta_{\text{int},a} = 2\pi V_{\text{int}}.$$

- For the external vertices, let $V_{\text{ext},C}$ denote the number of vertices of the triangulation that are also vertices of a boundary arc C_i.
- Let $V_{\text{ext},\text{T}}$ denote the number of external vertices of the triangulation that are not also vertices of any boundary arc C_i.
- Thus

$$V_{\text{ext}} = V_{\text{ext},C} + V_{\text{ext},T}.$$

Proof of Global Gauss-Bonnet Theorem

- Divide the external vertices of the triangulation into those from the arcs C_i and those from the triangulation alone so that

$$\sum_b \beta_{\mathrm{ext},b} = \sum_c \beta_{\mathrm{ext},C,c} + \sum_d \beta_{\mathrm{ext},T,d}.$$

- For vertices $\beta_{\mathrm{ext},T,d}$ of the triangulation but not of of an arc C_i, each vertex is a regular point of the curve C_i so that the sum of the two angles equals π. Thus

$$\sum_d \beta_{\mathrm{ext},T,d} = \pi V_{\mathrm{ext},T}.$$

- The remaining angles are *internal* angles at vertices of some C_i so that

$$\sum_c \beta_{\mathrm{ext},C,d} = \sum_{ij} \varphi_{ij} = \sum_{ij} \pi - \theta_{ij} = \pi V_{\mathrm{ext},C} - \sum_{ij} \theta_{ij}.$$

Proof of Global Gauss-Bonnet Theorem

- Thus we come to the end of the proof: we need to show

$$-\pi V_{\text{ext}} - \sum \beta_{nm} = -2\pi V + \sum \theta_{ij}.$$

- Summing up all our group of angles (internal, external and part of a C_i, external and not part of a C_i):

$$\begin{aligned}
-\pi V_{\text{ext}} - \sum \beta_{nm} &= -\pi V_{\text{ext}} - 2\pi V_{\text{int}} - \pi V_{\text{ext},T} - \left(\pi V_{\text{ext},C} - \sum_{ij} \theta_{ij}\right) \\
&= -\pi V_{\text{ext}} - \pi(V_{\text{ext},T} + \pi V_{\text{ext},C}) - 2\pi V_{\text{int}} + \sum_{ij} \theta_{ij} \\
&= -2\pi V_{\text{ext}} - 2\pi V_{\text{int}} + \sum_{ij} \theta_{ij} \\
&= -2\pi V + \sum_{ij} \theta_{ij}.
\end{aligned}$$